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OPTIMAL POLICIES FOR CAPACITY EXPANSION UNDER KNOWN DEMAND GROWTH

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ABSTRACT

This study is concerned with optimal policies for a sequence of discrete capacity expansions in response to expected growth in demand over an arbitrary planning interval. Demand growth is an arbitrary, differentiable, strictly increasing function of time. We assume economies of scale for investment costs. Time intervals between capacity expansions, as well as the magnitudes of the expansions, are continuous variables. Discounting is continuous.

The model used for a single capacity expansion is a member of an isomorphic family. The first model includes investment and shortage costs; the second includes investment, shortage and production costs; and, the last includes investment costs and revenue. The first model was chosen for analysis.

An asymptotic stationary model is derived for a linear demand function. This model assumes that capacity expansions are of equal size.

Optimal policies have the following property: either an expansion of size zero is optimal or capacity must be expanded to equal or exceed demand at the time of an expansion.

By detailed analysis of a single expansion problem, an algorithm is developed for determining both the optimal size and time of installation for the expansion. For a sequence of expansions over some planning horizon, a similar calculation is intractable. However, by taking discrete values for the size of expansion, which is reasonable, it is possible to use part of the algorithm for the single expansion problem to determine optimal sizes, times and number of expansions over any planning horizon.

Computations of optimal sizes, times and number of expansions are made for various demand curves. These computations show that the size and time of the first expansion (as a function of the planning interval) converge to fixed values. In the case of linear demand laws, it is shown that the general treatment used converges, as the planning interval tends to infinity, to the solution for a stationary model.

It is concluded that the models analyzed are susceptible to practical application, because the solution for the first expansion is not sensitive to the demand law used.

GLOSSARY OF FREQUENTLY USED SYMBOLS

$D(y)$ = Demand function

$F(\ )$ = Inverse function of D

I = Length of planning interval

v = Initial capacity at beginning of planning interval

x = Size of a capacity expansion

t = Time at which a capacity expansion occurs

n = Number of expansions

k = Investment cost constant

a = Economies of scale constant for investment costs

p = Penalty cost constant

r = Interest factor for present value computations

$$w = \left(\frac{rk}{p}\right)^{\frac{1}{1-a}}$$

$$z = \left[\frac{p}{rk}\,(D(0) - v)\right]^{\frac{1}{a}}$$

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CHAPTER I

INTRODUCTION

This paper is concerned with optimal policies for a sequence of discrete capacity expansions in response to expected growth in demand over an arbitrary time interval. Demand refers to a single commodity. For each expansion, there are two decision variables: the size of the expansion and the time at which it occurs. Capacity expansion over time is a step function. Discounting is continuous. The problem is analogous to a class of inventory problems in which order quantities are substituted for capacity increases.

Several investigators have studied aspects of the general problem of capacity increases. Since problems associated with continuous expansion have been widely studied, we mention here only a few references. Arrow, Beckman and Karlin [1] studied a model in which capacity may change continuously (positively or negatively) over time with an upper bound on the rate of change at any time. They assumed constant returns to scale for all operating and investment costs. Bradley [2] studied a multi-commodity continuous expansion problem.

Hinomoto [7] studied some effects of decreasing investment and operations costs over time which result from technological improvements.

Most of the other investigations assume: capacity expansion to be a step function; obsolescence of plant is omitted from consideration; lead time is not considered; economies of scale exist for plant investment costs. Following is a brief summary of the more recent studies.

Chenery [3] assumed no shortages, a constant rate of growth in demand and equal spacings between a finite number of capacity increases.

Howard and Nemhauser [8] consider a capacity expansion model analogous to an inventory model with demands known and capacity changes occurring at discrete points in time. They assume a composite convex function for the

combined shortage and excess capacity costs, hence economies of scale for investment costs are not explicitly considered. The model approximates the situation in which continuous capacity expansions are permitted. Manne [9] and [10] studied a stationary policy for a linear demand function over an infinite planning interval. He also considered a stationary model in which demand is a discrete random variable with an infinite planning interval.

Manne and Veinott [10] studied a model which assumed that capacity increases occur at given points in time, nondecreasing demand with shortage defined as the difference (positive) between demand and capacity only at the time of a capacity increase.

Srinivasan [10] studied a model which assumed no shortages and exponential demand growth over an infinite planning interval.

Erlenkotter [10] studied a model which assumed an infinite planning interval, constant growth rate of demand, and shortage defined as the difference (positive) between demand and capacity only at the time of a capacity increase.

The U. N. Bureau of Economic Affairs [11] published a study of a single expansion model, with piece-wise linear demand growth without discounting, in which an optimal policy is one which minimizes the investment cost per unit of output over a finite planning interval.

In this study, we allow demand to grow in an almost arbitrary way to account for the large variety of growth patterns which appear to be typical. For example, the world demand for nitrogenous fertilizer during 1955-1960 and cement during 1947-1960 grew at *decreasing* rates [12]. Hence a power function with an exponent less than one is a convenient way to describe demand as a function of time. A contrary condition prevails for glass container demand in Central America where demand in 1959 was growing *geometrically* [11]. It is not uncommon to observe demand growing at an

increasing rate at first but at a decreasing rate later. A representation of such a demand pattern takes the form of an "ogive," a curve often used to describe biological growth. Therefore, this study assumes that demand is a strictly increasing, differentiable function in order to account for a wide variety of growth situations. Monotonicity is required only for the uniqueness properties of the solutions. At the beginning of the planning interval, initial capacity may be larger or smaller than demand and initial demand is nonnegative.

An important assumption of most pre-investment studies is that investment costs are subject to economies of scale, e.g., double the capacity at less than double the cost. This is a wide spread phenomena. Haldi and Whitcomb [6] report on 221 cases of various kinds of industrial equipment in 14 countries, 84% of which were characterized by an economies of scale parameter of "a" which is less than one, with a modal value of 0.6. This seems to support the "two-thirds rule-of-thumb," commonly used in engineering studies. In United Nations [11], the values of 0.75 and 0.6 were used for a bottling plant and a fertilizer plant, respectively. Chilton [4] reviewed 36 types of chemical, petroleum and primary metal plants and concluded that a value of 0.6 is safe to use for pre-design economic studies. In this study, we assume that a facility of capacity "x" is constructed (or expanded) at a cost of kx^a where $k > 0$, $0 < a < 1$. This assumption was made by Manne [9] and [10], Srinivasan [10], Erlenkotter [10] and the United Nations Bureau of Economic Affairs [11].

Economies of scale for operating costs (excluding costs for capital) also occurs for many industries but appears to be less widespread and often less important than economies of scale for investments. For many industries, consumption of fuel, electric power, raw materials are almost proportional to production. Capital intensive industries, such as process industries, tend

to exhibit lower labor requirements per unit produced as plant size increases; however, for these industries labor cost is often small compared to capital cost. Accordingly, primary emphasis in this paper is given to economies of scale for investment costs.

Since the time intervals between expansions are continuous variables, we assume continuous compounding with an interest factor of "r" so, the present value of 1 expended t years, hence, amounts to e^{-rt}.

The number of increases in capacity, "n" is a parameter of significant interest. For a finite planning horizon, as "n" becomes large, the size of some, perhaps most, of the increases could become small and in any real situation, it may not be sensible to permit very small increases. Accordingly, one could set a lower bound on the size of a capacity increase, however, we omit this from explicit consideration.

In summary, the class of problems which are considered herein are characterized as follows:

(i) demand per unit time is a monotone increasing, continuously differentiable function of time.

(ii) capacity expansion is a step function.

(iii) shortages and excesses of capacity are permitted.

(iv) the time intervals between capacity expansions as well as the magnitudes of increases are *continuous* variables.

(v) discounting is continuous.

(vi) the planning interval is an arbitrary finite length of time. (An infinite planning interval is also included for a linear demand function.)

(vii) investment costs are subject to economies of scale.

(viii) cost components for optimization are the present values of investment cost and shortage cost or production cost or net revenue.

In Chapter II, we consider a family of single expansion models. The first model is concerned with optimal policies for investment and shortage costs--shortage cost arising as a result of penalties for supplying demand via some expensive alternative means such as renting operating facilities, or importing commodities. The second model does not penalize shortages directly but instead is concerned with optimal policies for investment cost and revenue produced by the installed capacity. The third model considers investment costs, shortage costs and production costs. These models are shown to be isomorphic to one another; consequently, only one is used for subsequent analysis and computation.

Chapter III considers in detail the single expansion model. The optimal time of expansion is considered as a function of the size of the expansion. Then, the optimal xize of expansion is considered. Finally, an algorithm is given for finding the optimal policy.

Chapter IV considers a sequence of expansions over time. A discrete dynamic programming algorithm is developed.

Chapter V considers an asymptotic stationary model for a linear demand function. It is shown to be a limiting case of a sequential expansion model of Chapter IV. The resulting equations are the same as found by Manne [9] and Erlenkotter [10] using quite a different approach.

In Chapter VI we present some computational results for single and sequential expansion problems among which is a direct comparison of some finite horizon expansion policies with an asymptotic stationary policy. Although the number of computations is limited, some interesting phenomena appear. The most interesting is that the size of the first expansion (as a function of the planning interval) and the time of installing the expansion converge to fixed values. Moreover, for a given set of parameters, the values are the same for a family of demand functions.

CHAPTER II

A FAMILY OF SINGLE EXPANSION MODELS

In this chapter we consider a family of models which characterize common investment situations. We assume shortages and excesses of capacity are permitted. For all models a single expansion of size "x" occurs at time "t" , $t \in [0,I]$ and the present value of the investment cost is $I(x,t)$ where:

$$I(x,t) = kx^a e^{-rt}$$

where "k" > 0 , is the cost per unit of capacity, "a" is a dimensionless scale factor, $0 < a < 1$, and "r" is the dimensionless interest factor.

Let "v" be the initial capacity at time zero and $D(t)$ be the demand at time t . Then $D(0)$ is the initial demand at time zero. We permit

$$D(0) \gtreqless v \ .$$

Let $F(\cdot)$ be the inverse function of $D(t)$ defined as follows:

$$F(v) = \min\{y \mid D(y) \geq v \ , \ y \in [0,I]\} \ .$$

From this it follows that:

$$F(D(y)) = y$$

$$D(F(v+x)) = v + x$$

$$F(v) = 0 \qquad \text{if} \quad v \leq D(0) \ .$$

Figure 1 shows these relationships for one of several possible configurations.

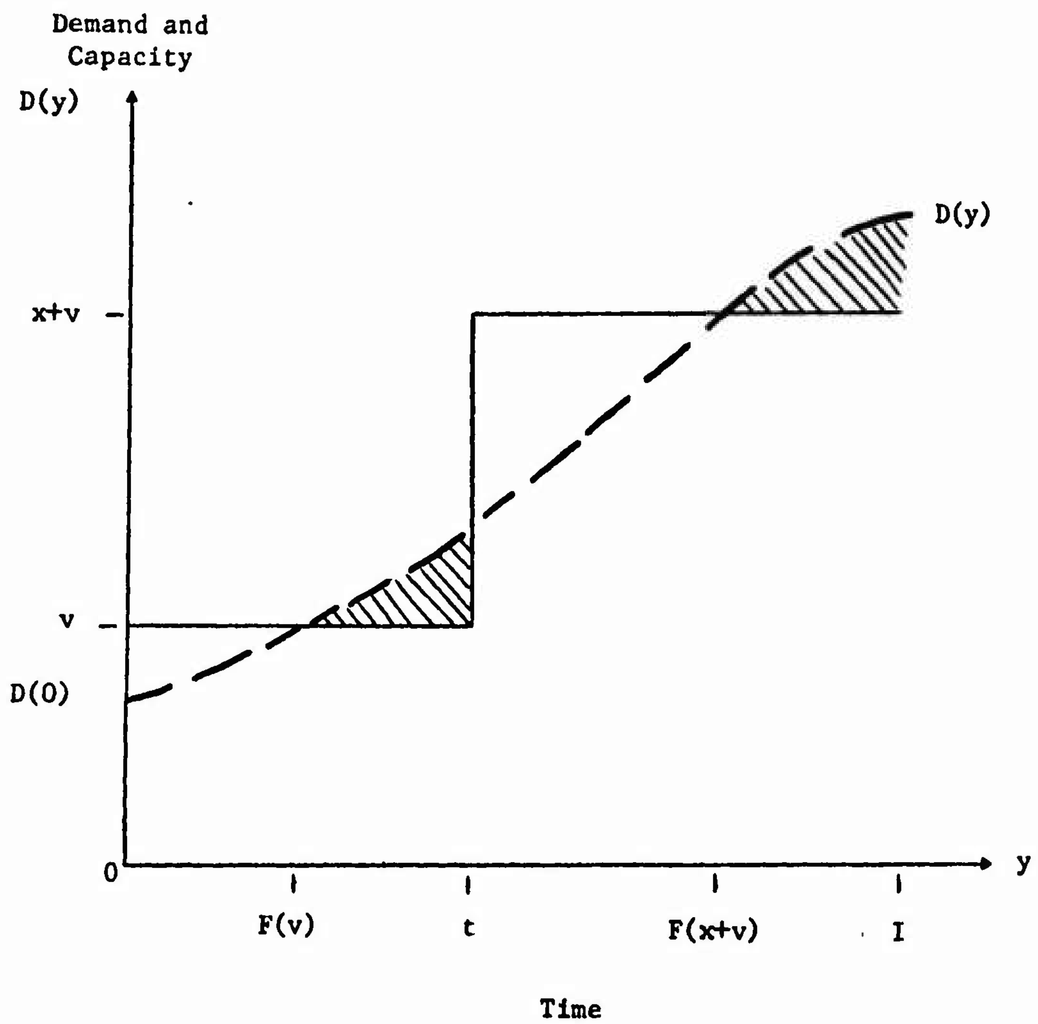

FIGURE 1

Define:

$$U(x,t) = \int_0^t \min\,(D(y),v)e^{-ry}dy$$

$$+ \int_t^I \min\,(D(y),v + x)e^{-ry}dy \;.$$

The general form for the present value of total cost for the family of single expansion models is:

$$J(x,t) = b_1 I(x,t) + b_2 U(x,t) + b_3$$

where b_1 , b_2 , b_3 are constants. We will illustrate three models which are useful in characterizing investment problems and which differ from one another only in the choice of parameters.

MODEL ONE

Assume capacity shortage at any time carries a penalty of $p per unit short. In Figure 1, the shortage is the shaded area between the demand and capacity curves. Let $T(x,t)$ be the present value of the total costs for investment and shortages. Then:

$$T(x,t) = I(x,t) - pU(x,t) + p \int_0^I D(y)e^{-ry}dy$$

hence

$$b_1 = 1 \;,\; b_2 = -p \;,\; b_3 = p \int_0^I D(y)e^{-ry}dy \;.$$

This first function has an alternate form which is more convenient to use,

namely:

$$T(x,t) = kx^a e^{-rt} + p \int_0^t \max\,[D(y) - v, 0] e^{-ry} dy$$

$$+ p \int_t^I \max\,[D(y) - v - x, 0] e^{-ry} dy \ .$$

MODEL TWO

Suppose revenue is earned in proportion to the output produced and that production is equal to the demand or capacity, which ever is smaller. For example, the cumulative output in Figure 1a corresponds to the shaded area. Assume production costs are proportional to output, hence revenue means net revenue. Let $\$q$ be the revenue per unit produced and $R(x,t)$ be the present value of the revenue less investment cost for an expansion of size x at time t over the planning period of length I. Then:

$$R(x,t) = -I(x,t) + qU(x,t) \ .$$

Model One and Model Two are simply related as follows:

$$T(x,t) = -R(x,t) + \text{constant} \ ,$$

whenever $p = q$. Here we may interpret shortage penalty as a loss in revenue.

MODEL THREE

Suppose that in addition to investment cost and shortage cost of Model One, we also incur a production cost of $\$c$ per unit produced. Assume as before that the amount produced is the minimum of capacity or demand. Let $P(x,t)$ be the present value of investment, shortage and production costs.

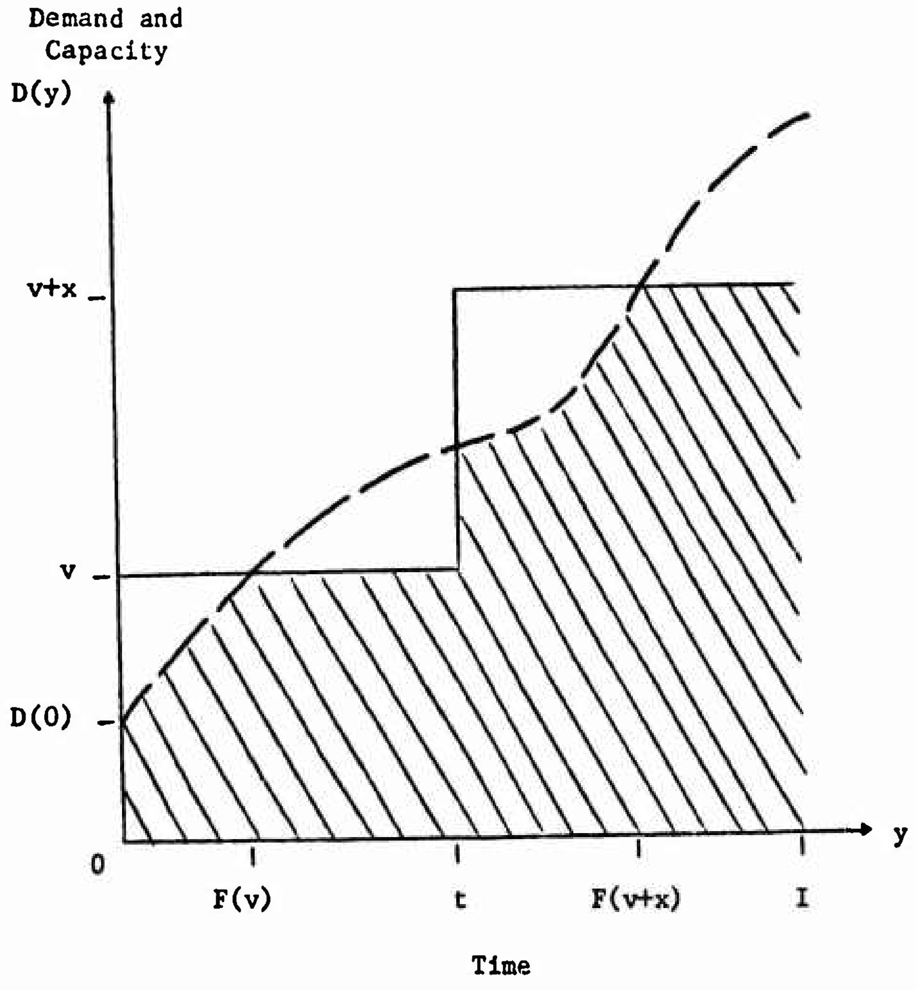

FIGURE 1a

Then:

$$P(x,t) = I(x,t) - (p - c)U(x,t) + p \int_0^I D(y)e^{-ry}dy \ .$$

This is of the same general form as $J(x,t)$ where

$$b_1 = 1 \ , \ b_2 = -(p - c) \ , \ b_3 = p \int_0^I D(y)e^{-ry}dy \ .$$

SUMMARY:

We have shown that three different models of investment situations belong to the general family of models defined by $J(x,t)$ and that they differ from each other only in the choice of parameter values. The properties of this family may be derived from or discussed in terms of any model in the family, in particular, the first model with cost function $T(x,t)$.

CHAPTER III

ANALYSIS OF THE SINGLE CAPACITY EXPANSION PROBLEM

COST AS A FUNCTION OF "t"

Consider the cost function $T(x,t)$. We will use the following abbreviated notation. Let $T(t \mid x)$ be the cost function for a given value of x and let

$$T'(t \mid x) = \frac{\partial}{\partial t} T(t,x)$$

with similar notations for other partial derivatives. Recall:

$$(1) \quad T(x,t) = kx^a e^{-rt} + p \int_0^t \max(D(y) - v, 0) e^{-ry} dy + p \int_t^I \max(D(y) - v - x, 0) e^{-ry} dy .$$

In general the first partial derivative with respect to t is:

$$(1a) \quad T'(t \mid x) = -rkx^a e^{-rt} + pe^{-rt} \max(D(t) - v, 0) - pe^{-rt} \max(D(t) - v - x, 0) .$$

Assume that $v < D(I)$ and that x is restricted so that $F(v + x) \leq I$. Hence $F(v) < I$. The relative magnitude of the initial conditions v and $D(0)$ provide two cases: $D(0) < v$ and $D(0) \geq v$. For each case there are three intervals for t for a given value of x.

Case A: $v > D(0)$

(i) $t \in [0, F(v)]$
or $D(t) - v \leq 0$

Hence: $T'(t \mid x) = -rkx^a e^{-rt} < 0$.

(ii) $\underline{t \in (F(v), F(v + x)]}$
or $D(t) - v > 0$

and $D(t) - v - x \leq 0$

Hence: $$T'(t \mid x) = e^{-rt}[-rkx^a + p(D(t) - v)] \ .$$

(iii) $\underline{t \in (F(v + x), I]}$
or $D(t) - v - x > 0$

Hence: $$T'(t \mid x) = e^{-rt}[-rkx^a + px] \ .$$

<u>Remark</u>:

$T'(t \mid x)$ is continuous for all $t \in [0,I]$ because

$$T'(t \mid x)\Big|_{t=F(v)} = -rkx^a e^{-rF(v)}$$

$$T'(t \mid x)\Big|_{t=F(v+x)} = (-rkx^a + px)e^{-rF(v+x)} \ .$$

Consider two intervals for x :

$$\text{A1:} \quad 0 \leq x \leq \left(\frac{rk}{p}\right)^{\frac{1}{1-a}}$$

$$\text{A2:} \quad \left(\frac{rk}{p}\right)^{\frac{1}{1-a}} < x \leq D(I) - v \ .$$

Define $w = \left(\frac{rk}{p}\right)^{\frac{1}{1-a}}$.

<u>Case A1</u>:

$$0 \leq x \leq w \Rightarrow px^{(1-a)} \leq rk \ ,$$

or

$$px \leq rkx^a \ .$$

Hence

$$T'(t \mid x) \leq 0 \qquad \text{for} \quad t \in [F(v + x), I) \, .$$

Now $[-rkx^a + p(D(t) - v]$ is strictly increasing in t for all $t \in [F(v), F(v + x)]$ hence

$$T'(t \mid x) < 0 \qquad \text{for all} \quad t \in [F(v), F(v + x)] \, .$$

Let $t^*(x)$ be the best choice of t for a given x. Then:

$$\underline{t^*(x) = I} \qquad \text{for} \quad 0 \leq x \leq \left(\frac{rk}{p}\right)^{\frac{1}{1-a}} = w \, .$$

<u>Case A2</u>:

The inequality of Case A2 implies:

$$px > rkx^a \, .$$

Hence

$$T'(t \mid x) > 0 \qquad \text{for} \quad t \in [F(v + x), I] \, .$$

But from (i) we have

$$T'(t \mid x) < 0 \qquad \text{for} \quad t \in [0, F(v)] \, .$$

Hence $T'(t \mid x)$ changes sign from negative to positive for $t \in (F(v), F(v + x))$ and since $[-rkx^a + p(D(t) - v)]$ is strictly increasing in t it follows that there exists a unique $t^*(x)$ satisfying

$$D(F(v)) < D(t^*(x)) = \left(\frac{rk}{p}\right)x^a + v < D(F(v + x))$$

and

$$\underline{t^*(x) = F\left(\frac{rk}{p} x^a + v\right)} \qquad \text{for} \quad w < x \leq D(I) - v \, .$$

Summary of Case A: $v > D(0)$

$$t^*(x) = \begin{cases} I & \text{for } 0 \leq x \leq \min\{w, D(I) - v\} \\ F\left(\frac{rk}{p} x^a + v\right) & \text{for } w < x \leq D(I) - v \end{cases} .$$

Note: for $t^*(x) < I$ we have:

$$(2) \qquad F(v + w) < F(v + w^{1-a}x^a) < F(v + x) .$$

Case B: $0 \leq v \leq D(0)$

Denote $z = \left[\frac{p}{rk}(D(0) - v)\right]^{\frac{1}{a}}$. Here we must consider several intervals for x depending on the relative size of w and $D(0) - v$ as follows.

B(1)a: $0 \leq x \leq w < D(0) - v$

B(1)b: $0 < w < x \leq D(0) - v$

B(1)c: $0 < w < D(0) - v < x \leq z$

B(1)d: $0 < w < D(0) - v < z < x \leq D(I) - v$

B(2)a: $0 \leq x \leq D(0) - v \leq w$

B(2)b: $D(0) - v < x \leq w$

B(2)c: $D(0) - v \leq w < x \leq D(I) - v$

Note: Cases B(1)a through B(1)d assume

$$w < D(0) - v$$

whereas Cases B(2)a through B(2)c assume

$$w \geq D(0) - v .$$

Note: $0 \leq v \leq D(0) \Rightarrow F(v) = 0$ hence for B(1)a and B(1)b:

$$x \leq D(0) - v \Rightarrow \begin{Bmatrix} D(t) - v - x \geq 0 \\ D(t) - v \geq 0 \end{Bmatrix} \qquad \forall\ t \in [0,I]$$

hence $T'(t \mid x) = e^{-rt}[-rkx^a + px]$ for all t in the interval $t \in [0,I]$. This single form for $t'(t \mid x)$ simplifies the analysis for Cases B(1)a and B(1)b.

Case B(1)a:

$$\begin{aligned} x \leq w &\Rightarrow px^{1-a} \leq rk \\ &\Rightarrow px \leq rkx^a \\ &\Rightarrow -rkx^a + px \leq 0 \end{aligned}$$

thus Equation (1a) is nonpositive, i.e.,

$$T'(t \mid x) \leq 0 \qquad \forall\ t \in [0,I]$$

hence

$$\underline{t^*(x) = I} \qquad \text{for} \quad 0 \leq x \leq w \ .$$

Case B(1)b:

$$w < x \leq D(0) - v \ .$$

This implies that $-rk + px^{1-a} > 0$ or

$$-rkx^a + px > 0 \ ,$$

thus Equation (1a) is positive, i.e.,

$$T'(t \mid x) > 0 \qquad \forall\ t \in [0,I] \ .$$

Hence

$$\underline{t^*(x) = 0} \qquad \text{for} \quad w < x \leq D(0) - v \ .$$

Summary for Case B(1)a and B(1)b

$$t^*(x) = \begin{cases} I & \text{for} \quad 0 \leq x \leq w \leq D(0) - v \\ 0 & \text{for} \quad 0 < w < x \leq D(0) - v \end{cases} .$$

<u>Case B(1)c</u>:

(2a) $$0 < w < D(0) - v < x \leq z .$$

Consider the two intervals on t :

(i) $t \in (F(v + x), I]$

$$T'(t \mid x) = e^{-rt}[-rkx^a + px]$$

(ii) $t \in [0, F(v + x)]$

$$T'(t \mid x) = e^{-rt}[-rkx^a + p(D(t) - v)] .$$

Now (2a) implies $-rk + px^{1-a} > 0$ or

$$-rkx^a + px > 0 .$$

Hence

(3) $$T'(t \mid x) > 0 \qquad \forall\, t \in (F(v + x), I] .$$

Note, however, that $[-rkx^a + p(D(t) - v)]$ is strictly increasing in t and $-rkx^a + p(D(0) - v)$ may be nonnegative (when $t = 0$) . But Case B(1)c implies:

$$\left(\frac{rk}{p}\right)^{\frac{1}{1-a}} < D(0) - v \Rightarrow \frac{rk}{p} < (D(0) - v)^{1-a}$$

$$\Rightarrow (D(0) - v)^a < \frac{p}{rk}(D(0) - v)$$

$$\Rightarrow D(0) - v < \left[\frac{p}{rk}(D(0) - v)\right]^{\frac{1}{a}} .$$

Hence for $D(0) - v < x \leqq \left[\frac{p}{rk}(D(0) - v)\right]^{\frac{1}{a}}$, Case B(1)c means that

$$rkx^a \leqq p(D(0) - v) \Rightarrow -rkx^a + p(D(0) - v) \geqq 0$$

$$\Rightarrow T'(t \mid x) \geqq 0 \qquad \text{at} \quad t = 0 .$$

Since $T'(t \mid x)$ is strictly increasing this implies

(4) $$T'(t \mid x) \geqq 0 \qquad \forall\, t \in [0,F(v + x)] .$$

Inequalities (3) and (4) together imply that

$$T'(t \mid x) \geqq 0 \qquad \forall\, t \in [0,I] .$$

Hence: $\underline{t^*(x) = 0}$ for Case B(1)c.

<u>Remark</u>:

Whenever $\left[\frac{p}{rk}(D(0) - v)\right]^{\frac{1}{a}} \geqq D(I) - v$ Case B(1)c can be written:

$$w < D(0) - v < x \leqq D(I) - v$$

because we have restricted $x \in [0,D(I) - v]$.

<u>Case B(1)d</u>:

$$0 < w < D(0) - v < z < x \leqq D(I) - v .$$

For the two intervals on t :

(i) $t \in (F(v+x), I]$

$$T'(t \mid x) = e^{-rt}[-rkx^a + px]$$

(ii) $t \in [0, F(v+x)]$

$$T'(t \mid x) = e^{-rt}[-rkx^a + p(D(t) - v)] \ .$$

Case B(1)d implies

$$rk < px^{1-a}$$

$$rkx^a < px$$

$$-rkx^a + px > 0 \Rightarrow T'(t \mid x) > 0 \qquad \text{for } t \in [F(v+x), I] \ .$$

But Case B(1)d also implies

$$\frac{p}{rk}(D(0) - v) < x^a \Rightarrow p(D(0) - v) < rkx^a$$

$$\Rightarrow -rkx^a + p(D(0) - v) < 0$$

$$\Rightarrow T'(t \mid x)\Big|_{t=0} < 0 \ .$$

Since $[-rkx^a + p(D(t) - v)]$ is strictly increasing there exists a value $t^*(x) \in (0, F(v+x))$ such that $T'(t^*(x) \mid x) = 0$ and $t^*(x)$ satisfies

$$D(t^*(x)) = \frac{rk}{p} x^a + v < (v + x)$$

or

(5) $$t^*(x) = F\left(\frac{rk}{p} x^a + v\right) < F(v + x)$$

for Case B(1)d. Summary for $w < D(0) - v$ and $0 \leq v \leq D(0)$:

$$t^{*}(x) = \begin{Bmatrix} I & \text{for} \quad 0 \leqq x \leqq w \\ 0 & \text{for} \quad w < x \leqq \min\{z, D(I) - v\} \\ F(v + w^{1-a}x^{a}) & \text{for} \quad z < x \leqq (D(I) - v) \end{Bmatrix} .$$

<u>Case B(2)a</u>:

$$0 \leqq x \leqq D(0) - v \leqq w .$$

This implies that $-rkx^{a} + px \leqq 0$, and since $T'(t \mid x) = e^{-rt}(-rkx^{a} + px)$ for all $t \in [0,I]$,

$$T'(t \mid x) \leqq 0 .$$

Hence <u>$t^{*}(x) = I$</u> for Case B(2)a.

<u>Case B(2)b</u>:

$$D(0) - v < x \leqq w .$$

Consider the intervals on t :

(i) $t \in [0,F(v + x)]$

$$T'(t \mid x) = e^{-rt}(-rkx^{a} + p(D(t) - v))$$

(ii) $t \in (F(v + x),I]$

$$T'(t \mid x) = e^{-rt}(-rkx^{a} + px) .$$

Case B(2)b implies

$$px^{1-a} \leqq rk \Rightarrow px \leqq rkx^{a}$$

$$\Rightarrow -rkx^{a} + px \leqq 0$$

hence $T'(t \mid x) \leqq 0$ for $t \in (F(v + x),I]$. But Case B(2)b also implies

$$p(D(0) - v) < px \Rightarrow -rkx^a + p(D(0) - v) < -rkx^a + px \leq 0$$

hence

$$T'(t \mid x)\Big|_{t=0} < 0 .$$

Since $[-rkx^a + p(D(t) - v)]$ is increasing in t , and since $T'(t \mid x) \leq 0$ at $t = 0$ as well as $t \in (F(v + x), I]$, it follows that

$$T'(t \mid x) \leq 0 \qquad \forall\, t \in [0,I] .$$

Hence $\underline{t^*(x) = I}$ for Case B(2)b.

<u>Case B(2)c</u>:

$$D(0) - v \leq w < x \leq D(I) - v ,$$

or

$$-rkx^a + px > 0 .$$

Hence

$$T'(t \mid x) > 0 \qquad \forall\, t \in (F(v + x), I] .$$

Now, since $x > (D(0) - v)$,

$$T'(t \mid x)\Big|_{t=0} < [-rk(D(0) - v)^a + p(D(0) - v)]e^{-rt} .$$

But Case B(2)c implies

$$D(0) - v \leq w ,$$

or

$$p(D(0) - v) \leq rk(D(0) - v)^a .$$

Hence

$$T'(t \mid x)\Big|_{t=0} < 0 .$$

Since $[-rkx^a + p(D(t) - v)]$ is strictly increasing in t there exists a value $t^*(x) \in (0, F(v + x))$ such that $D(t^*(x)) = \left(\frac{rk}{p}\right)x^a + v < (v + x)$ or

$$t^*(x) = F\left(\frac{rk}{p} x^a + v\right) < F(v + x) \tag{6}$$

for Case B(2)c. Summary for B(2)a, B(2)b, and B(2)c, i.e.,

$$D(0) - v \leq w < D(I) - v \quad \text{and} \quad 0 \leq v \leq D(0) :$$

$$t^*(x) = \begin{Bmatrix} I & \text{for} \quad 0 \leq x \leq w \\ F\left(\frac{rk}{p} x^a + v\right) & \text{for} \quad w < x \leq D(I) - v \end{Bmatrix} .$$

<u>Summary for all cases</u>:

(i) <u>$t^*(x) = I$</u> for:

$$0 \leq x \leq \min \{w, (D(I) - v)\}$$

(ii) <u>$t^*(x) = 0$</u> for:

$$\min \{w, D(I) - v\} < x \leq \min \left\{ \max \begin{Bmatrix} \max\, [D(0) - v, z] \\ \min\, [w, (D(I) - v)] \end{Bmatrix} , D(I) - v \right\} .$$

(iii) <u>$t^*(x) = F\left(\frac{rk}{p} x^a + v\right)$</u> for:

$$\min \left\{ \max \begin{Bmatrix} \max\, [D(0) - v, z] \\ \min\, [w, D(I) - v] \end{Bmatrix} , D(I) - v \right\} < x \leq (D(I) - v) .$$

These hold for both initial conditions, $0 \leqq v \leqq D(0)$ and $0 \leqq D(0) < v$, and they may be used to set up computer codes.

Graphs of $t^*(x)$:

There are four possible situations for $t^*(x)$ as follows. See Figures 2, 3, 4, and 5. (These are drawn left continuous.)

(i) $w > D(I) - v$

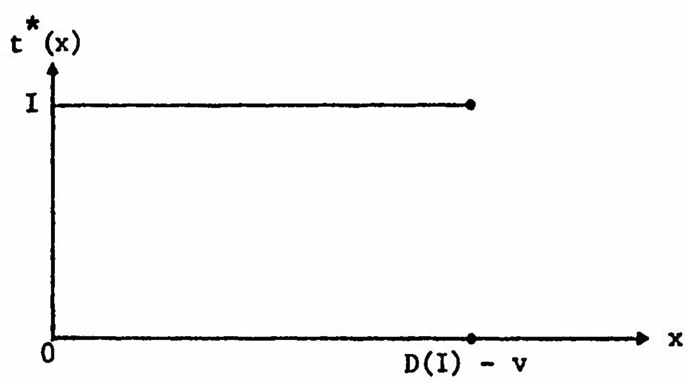

FIGURE 2

(ii) $(D(0) - v) \leqq w < (D(I) - v)$

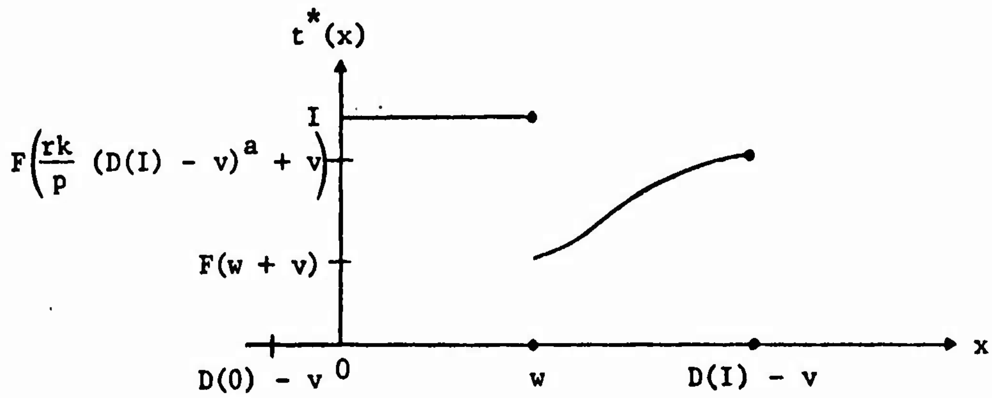

FIGURE 3

Note:

$$F(w^{1-a}x^a + v)\Big|_{x=w} = F(v + w) .$$

Note:

$$F\left(\frac{rk}{p} (D(I) - v)^a + v\right) < I .$$

The strict inequality holds because otherwise $(D(I) - v) = w$ which violates this case.

(iii) $w < D(0) - v$ and $z < (D(I) - v)$

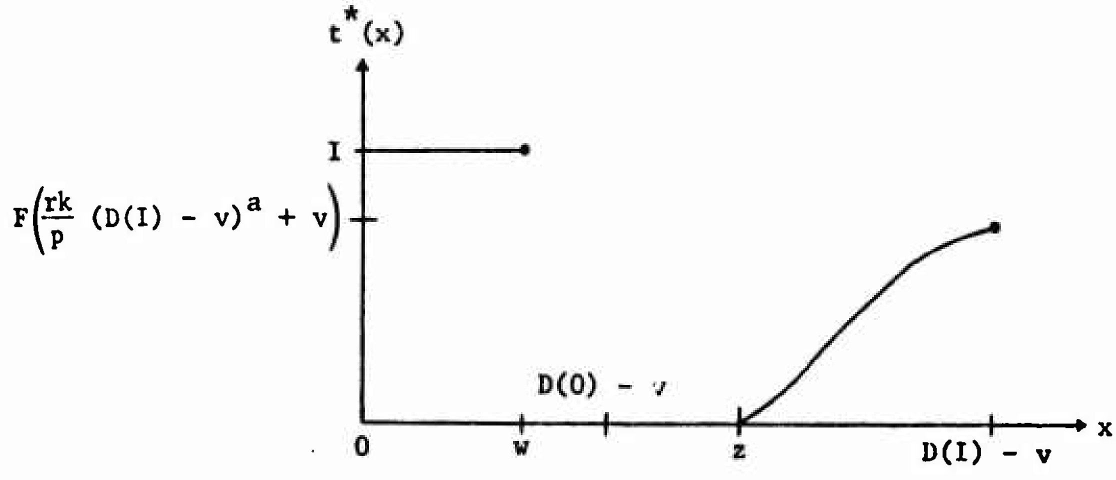

FIGURE 4

(iv) $w < D(0) - v$ and $z \geq D(I) - v$

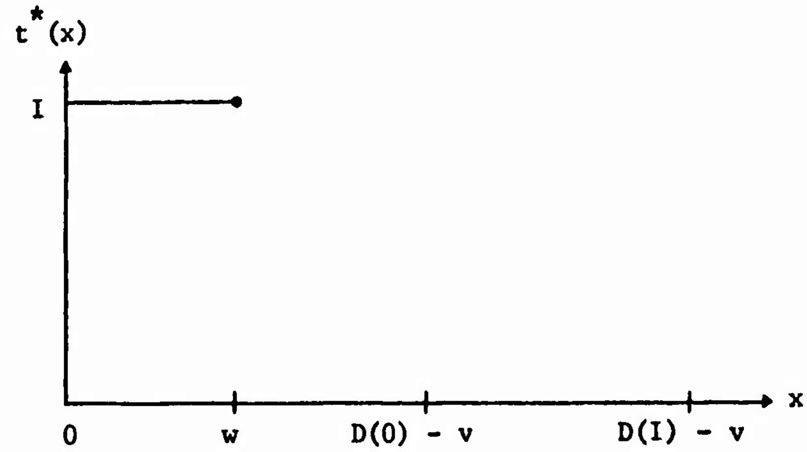

FIGURE 5

Remarks:

Recall from (2), (5) and (6) that whenever $t^*(x) = F(w^{1-a}x^a + v)$ we have that

$$t^*(x) < F(v + x)$$

regardless of initial conditions $v > D(0)$ or $v \leq D(0)$. Hence the graphs of these configurations are as follows.

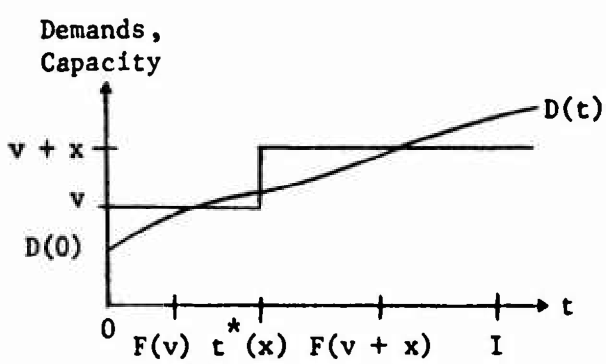

FIGURE 6a

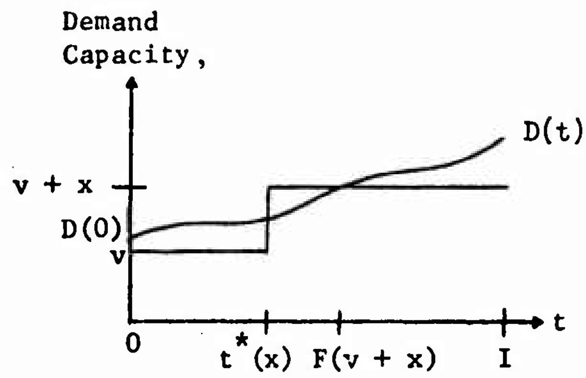

FIGURE 6b

OPTIMAL x

Let $T(x) = T(t^*(x),x)$. There are several cases to examine as follows.

I: $w > D(I) - v$

II: $D(0) - v \leq w \leq D(I) - v$

III: $w < D(0) - v$ and $z < D(I) - v$

IV: $w < D(0) - v$ and $z \geq D(I) - v$

Case I: $w > D(I) - v$; $t^*(x) = I$.

$$T(x) = kx^a e^{-rI} + p \int_0^I \max\,(D(y) - v,0)e^{-ry}dy$$

$$T'(x) = akx^{a-1}e^{-rI} > 0 \qquad \forall\ x \in (0,D(I) - v] .$$

Hence $\min T(x) = T(0)$.

Case II: $D(0) - v \leq w \leq D(I) - v$.

Case II(a): $0 \leq x \leq w$; $t^*(x) = I$

$$T(x) = kx^a e^{-rI} + p \int_0^I \max\,(D(y) - v,0)e^{-ry}dy$$

$$T'(x) = akx^{a-1}e^{-rI} > 0 \qquad \forall\, x \in (0,w] .$$

Hence $T(x)$ is increasing for all $x \in (0,w]$.

Case II(b): $w < x < (D(I) - v)$; $t^*(x) = F(v + w^{1-a}x^a) < F(v + x)$.

$$T(x) = kx^a e^{-rt^*(t)} + p \int_0^{t^*(x)} \max\,(D(y) - v, 0)e^{-ry}dy$$

$$+ p \int_{t^*(x)}^{I} \max\,(D(y) - v - x, 0)e^{-ry}dy .$$

Referring to Figure 6, we note that $t^*(x) < F(v + x)$, $y \in [0,F(v)) \Rightarrow D(y) - v \leq 0$ and $y \in [0,F(v + x)] \Rightarrow D(y) - v - x \leq 0$. Hence for the range $w < x < D(I) - v$ we have:

$$\int_0^{t^*(x)} \max\,(D(y) - v, 0)e^{-ry}dy = \int_{F(v)}^{t^*(x)} (D(y) - v)e^{-ry}dy$$

$$\int_{t^*(x)}^{I} \max\,(D(y) - v - x, 0)e^{-ry}dy = \int_{F(v+x)}^{I} (D(y) - v - x)e^{-ry}dy .$$

Hence the cost function may be written:

$$T(x) = kx^a e^{-rt^*(x)} + p \int_{F(v)}^{t^*(x)} (D(y) - v)e^{-ry}dy$$

$$+ p \int_{F(v+x)}^{I} (D(y) - v - x)e^{-ry}dy .$$

Hence: $T'(x) = akx^{a-1}e^{-rt^*(x)} - \frac{p}{r}\,(e^{-rF(v+x)} - e^{-rI})$.

Denote: $T(w+0) = T(x)\Big|_{x=w+0}$; $T(0) = T(x)\Big|_{x=0}$. Then compare $T(w+0)$ with $T(0)$, where:

$$T(0) = p \int_{F(v)}^{I} (D(y) - v)e^{-ry}dy \ ,$$

since $y \in [0,F(v)] \Rightarrow D(y) \leq v$ And

$$T(w+0) = kw^a e^{-rF(v+w)} + p \int_{F(v)}^{I} (D(y) - v)e^{-ry}dy$$

$$- pw \int_{F(v+w+0)}^{I} e^{-ry}dy \ ,$$

because at $x = w + 0$, $t^*(x+0) = F(v+w+0)$. Hence:

$$T(w+0) - T(0) = kw^a e^{-rF(w+v)} + \frac{pw}{r}(e^{-rI} - e^{-rF(v+w)})$$

$$= \left(kw^a - \frac{pw}{r}\right)e^{-rF(v+w)} + \frac{pw}{r}e^{-rI} \ .$$

But

$$kw^a = \frac{pw}{r} \ ,$$

hence

$$T(w+0) - T(0) = \frac{pw}{r}e^{-rI} > 0 \ ,$$

or

$$\underline{T(w+0) > T(0)} \ .$$

Case IIb: Conditions for $T'(x) \geq 0$

Recall that for $x \in (w, D(I) - v]$ we found that

$$F(v + w) < t^*(x) = F(v + w^{1-a}x^a) < F(v + x)$$

and

$$F(v + w^{1-a}x^a) \leq F(v + w^{1-a}(D(I) - v)^a) < I \ .$$

Then we can write

$$e^{-rF(v+w)} > e^{-rt^*(x)} \geq e^{-rF(v+w^{1-a}(D(I)-v)^a)} > e^{-rI} \ .$$

And since x^{a-1} is decreasing in x ,

$$\frac{p}{r} = kw^{a-1} > kx^{a-1} \geq k(D(I) - v)^{a-1} \ .$$

We now use these results to find a lower bound on the derivative for $x \in (w, D(I) - v]$ as follows:

$$T'(x) \geq ak(D(I) - v)^{a-1}e^{-rF(v+w^{1-a}(D(I)-v)^a)} + \frac{p}{r}(e^{-rI} - e^{-rF(v+w)}) \ ,$$

which is nonnegative if

$$a\left(\frac{w}{D(I) - v}\right)^{1-a} e^{-rF(v+w^{1-a}(D(I)-v)^a)} + (e^{-rI} - e^{-rF(v+w)}) \geq 0$$

or

$$(7) \quad I \leq -\frac{1}{r} \ln \left[e^{-rF(v+w)} - a\left(\frac{w}{D(I) - v}\right)^{1-a} e^{-rF(v+w^{1-a}(D(I)-v)^a)}\right] \ ,$$

in which case $T(x)$ is nondecreasing throughout the interval $(w, D(I) - v]$.

An example:

The importance of this result may be seen by means of an illustration. Suppose $D(y)$ is linear and units of time are chosen so that $D(y) = y$. Then

$$F(v + w) = v + w .$$

Suppose $v = 0$, $D(0) = 0$, $a = \frac{1}{2}$, $r = 0.1$, $k = 20$, $p = 1$, then $w = \left(\frac{rk}{p}\right)^{\frac{1}{1-a}} = 4$. Then (7), the condition on I becomes:

$$I \leqq -10 \ln \left\{ e^{-.4} - \frac{1}{\sqrt{I}} e^{-.2\sqrt{I}} \right\}$$

or

$$I \leqq 7.7 \text{ units of time approximately .}$$

Note: r corresponds to a bank interest rate of 6% , i.e.,

$$1.06 = \int_0^1 e^{ry} dy = \frac{e^r - 1}{r}$$

and

$$r = .1 \text{ approximately .}$$

Hence, for this example, $I \leqq 7.7$ units of time means $x^* = 0$ is the optimal policy.

$T(x)$ is not strictly decreasing for all $x \in (w, D(I) - v]$ because

$$T'(x)\Big|_{x=D(I)-v} = \left[ak(D(I) - v)^{a-1} e^{-rF(v+w^{1-a}(D(I)-v)^a)}\right] > 0 .$$

Furthermore, a sufficient condition for the existence of relative minima for some $x \in (w, D(I) - v)$ is

$$\text{(7a)} \qquad I > \frac{1}{r} \ln \left[\frac{e^{rF(v+w)}}{1-a} \right] .$$

which implies that

$$T'(x)\Big|_{x=w+0} = akw^{a-1}e^{-rF(v+w)} - \frac{p}{r}\left(e^{-rF(v+w)} - e^{-rI}\right) < 0 .$$

But if neither (7) nor (7a) holds, there still may be relative minima for some $x \in (w, D(I) - v]$.

Let x_o be the solution to

$$a\left(\frac{w}{x}\right)^{1-a} e^{-rF(v+w^{1-a}x^{a})} = e^{-rF(v+x)} - e^{-rI}$$

which yields the smallest value of $T(x)$ over the interval $(w, D(I) - v]$. Denote

$$T(x_o) = T(x)\Big|_{x=x_o}$$

$$T(0) = T(x)\Big|_{x=0} .$$

Then x_o is not the optimal value of x unless

$$T(x_o) < T(0) .$$

<u>Case II(b): Compare $T(x_o)$ with $T(0)$.</u>

$$T(x_o) = kx_o^a e^{-rt^*(x_o)} + p \int_{F(v)}^{t^*(x_o)} (D(y) - v)e^{-ry}dy$$

$$+ p \int_{F(v+x_o)}^{I} (D(y) - v - x_o)e^{-ry}dy$$

$$= kx_o^a e^{-rt^*(x_o)} + T(t^*(0),0) - px_o \int_{F(v+x_o)}^{I} e^{-ry}dy$$

$$- p \int_{t^*(x_o)}^{F(v+x_o)} (D(y) - v)e^{-ry}dy \ .$$

The integrand of the last term has known bounds, because for

$$t^*(x_o) = F\left(v + w^{1-a}x_o^a\right) \leqq y \leqq F(v + x_o) \ ,$$

$$v + w^{1-a}x_o^a \leqq D(y) \leqq v + x_o$$

and

$$w^{1-a}x_o^a \leqq D(y) - v \leqq x_o \ .$$

Define $\Delta = T(x_o) - T(0)$. Then

$$\left\{\begin{aligned} kx_o^a e^{-rt^*(x_o)} &+ kx_o^a\left(\frac{x_o}{w}\right)^{1-a}\left(e^{-rI} - e^{-rF(v+x_o)}\right) \\ &+ kx_o^a\left(\frac{x_o}{w}\right)^{1-a}\left(e^{-rF(v+x_o)} - e^{-rt^*(x_o)}\right)\end{aligned}\right\} \leqq \Delta \ ,$$

$$\left\{\begin{aligned} kx_o^a e^{-rt^*(x_o)} &+ kx_o^a\left(\frac{x_o}{w}\right)^{1-a}\left(e^{-rI} - e^{-rF(v+x_o)}\right) \\ &+ kx_o^a\left(e^{-rF(v+x_o)} - e^{-rt^*(x_o)}\right) \end{aligned}\right\} \gtreqless \Delta ,$$

$$kx_o^a\left\{\left[1 - \left(\frac{x_o}{w}\right)^{1-a}\right]e^{-rt^*(x_o)} + \left(\frac{x_o}{w}\right)^{1-a} e^{-rI}\right\} \leqq \Delta ,$$

and

$$kx_o^a\left\{\left[1 - \left(\frac{x_o}{w}\right)^{1-a}\right]e^{-rF(v+x_o)} + \left(\frac{x_o}{w}\right)^{1-a} e^{-rI}\right\} \geqq \Delta .$$

Then

$$\left(\frac{x_o}{w}\right)^{1-a} e^{-rI} < \left[\left(\frac{x_o}{w}\right)^{1-a} - 1\right]e^{-rF(v+x_o)} \Rightarrow \Delta < 0 ,$$

and

$$\left(\frac{x_o}{w}\right)^{1-a} e^{-rI} \geqq \left[\left(\frac{x_o}{w}\right)^{1-a} - 1\right]e^{-rt^*(x_o)} \Rightarrow \Delta \geqq 0 .$$

Hence

$$\left\{\begin{aligned} x^* &= x_o \quad \text{if} \quad I > F(v + x_o) + \frac{1}{r}\left[\ln\left(\frac{x_o}{w}\right)^{1-a} - \ln\left[\left(\frac{x_o}{w}\right)^{1-a} - 1\right]\right] \\ x^* &= 0 \quad \text{if} \quad I \leqq F\left(v + w^{1-a}x_o^a\right) + \frac{1}{r}\left[\ln\left(\frac{x_o}{w}\right)^{1-a} - \ln\left[\left(\frac{x_o}{w}\right)^{1-a} - 1\right]\right] \end{aligned}\right\} .$$

But if

$$F\left(v + w^{1-a}x_o^a\right) + \frac{1}{r}\,\ln\left[\frac{1}{1-\left(\frac{w}{x_o}\right)^{1-a}}\right] < I \leqq F(v + x_o) + \frac{1}{r}\,\ln\left[\frac{1}{1-\left(\frac{w}{x_o}\right)^{1-a}}\right] ,$$

then the role of x_o is uncertain and $T(x_o)$ must be compared directly with $T(0)$.

<u>Case III: $w < D(0) - v$ and $z < D(I) - v$</u> .

(See Figure 4). Consider four intervals: (a) $x \in [0,w]$, (b) $x \in (w,D(0) - v]$, (c) $x \in (D(0) - v,z]$, (d) $x \in (z,D(I) - v]$.

<u>Case III(a): $x \in [0,w]$; $t^*(x) = I$</u>

$$T(x) = kx^a e^{-rI} + p \int_0^I \max\,(D(y) - v,0)e^{-ry}dy$$

$$T'(x) = akx^{a-1}e^{-rI} > 0 \qquad \forall\ x \in (0,w]$$

$$T''(x) = (a - 1)akx^{a-2}e^{-rI} < 0 \qquad \forall\ x \in (0,w] .$$

Here $T(x)$ is concave increasing in this interval.

<u>Case III(b): $x \in (w,D(0) - v]$; $t^*(x) = 0$</u>

Note that $\max\,(D(y) - v - x,0) = D(y) - v - x$, and

$$T(x) = kx^a + p \int_0^I [D(y) - v - x]e^{-ry}dy .$$

Hence

$$T'(x) = akx^{a-1} + \frac{p}{r}\,(e^{-rI} - 1) .$$

and

$$T''(x) = a(a-1)kx^{a-2} < 0 .$$

Hence $T(x)$ is concave for all $x \varepsilon (w,D(0)-v]$, but the sign of $T'(x)$ is uncertain.

Note that

$$T'(x)\Big|_{x=w} = akw^{a-1}e^{-rI} ,$$

and

$$T'(x)\Big|_{x=w+0} = akw^{a-1} + \frac{p}{r}(e^{-rI} - 1) .$$

Hence:

$$\begin{aligned} T'(x)\Big|_{x=w+0} - T'(x)\Big|_{x=w} &= akw^{a-1}[1 - e^{-rI}] - \frac{p}{r}(1 - e^{-rI}) \\ &= \left(akw^{a-1} - \frac{p}{r}\right)(1 - e^{-rI}) \\ &= (a-1)kw^{a-1}(1 - e^{-rI}) < 0 . \end{aligned}$$

Therefore $T(x)$ is concave for all $x \varepsilon [0,D(0)-v]$ and if relative minima exist in the interval $x \varepsilon (w,z]$ they can only occur in the interval $x \varepsilon (D(0)-v,z]$.

Case I⁻⁻(c): $x \varepsilon (D(0)-v,z]$; $t^*(x) = 0$

$$T(x) = kx^a + p\int_{F(v+x)}^{I} (D(y) - v - x)e^{-ry}dy .$$

(Note: $T(x)$ is continuous at $x = D(0) - v$.)

$$T'(x) = akx^{a-1} + \frac{p}{r}(e^{-rI} - e^{-rF(v+x)}) .$$

$$T''(x) = a(a-1)kx^{a-2} + p(e^{-rF(v+x)})\left(\frac{d}{dx}F(v+x)\right) ,$$

and the signs of both $T'(x)$ and $T''(x)$ are uncertain. (Note that $T'(x)$ is continuous at $x = D(0) - v$.)

<u>Case III(b): Conditions for $T'(x) \geqq 0$ and $T(x) < 0$ for $x \in (w,z]$</u>

Recall that for Case III, we have that $w + v < D(0)$ and $F(v + w + 0) = 0$. And, since:

$$T'(x) = akx^{a-1} + \frac{p}{r} e^{-rI} - \frac{p}{r} e^{-rF(v+x)} ,$$

we can write:

$$1 = e^{-rF(v+w+0)} > e^{-rF(v+x)} \geqq e^{-rF(v+z)} .$$

Also recall that for the interval $x \in (w,z]$

$$\frac{p}{r} = kw^{a-1} > kx^{a-1} \geqq kz^{a-1} .$$

Thus we may write bounds on the derivative as follows.

$$T'(x) < akw^{a-1} - \frac{p}{r}(e^{-rF(v+z)} - e^{-rI}) ,$$

the right side of this inequality is negative whenever

$$e^{-rI} < (e^{-rF(v+z)} - a) ,$$

or

$$\text{(8)} \qquad I > \frac{1}{r} \ln\left(\frac{1}{e^{-rF(v+z)} - a}\right) .$$

Further, we can write an upper bound on $T'(x)$ as follows:

$$T'(x) \geq akz^{a-1} - \frac{p}{r}\left(e^{-rF(v+w+0)} - e^{-rI}\right)$$

or

$$T'(x) \geq akz^{a-1} - \frac{p}{r}\left(1 - e^{-rI}\right)$$

because $F(v + w + 0) = 0$. The right side of this inequality is positive whenever

$$a\left(\frac{w}{z}\right)^{1-a} \geq 1 - e^{-rI} ,$$

or

(8a) $$I \leq \frac{1}{r}\,\ell n\left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right) .$$

If neither (8) nor (8a) holds then (9) holds, where

(9) $$\frac{1}{r}\,\ell n\left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right) < I \leq \frac{1}{r}\,\ell n\left(\frac{1}{e^{-rF(v+z)} - a}\right) ,$$

and the sign of $T'(x)$ for $x \in (w,z]$ is uncertain, in which case there may exist relative minima in the interval $(D(0) - v,z]$. Note that sufficient conditions for relative minima for $x \in (D(0) - v,z]$ are that

$$T'(x)\Big|_{x=D(0)-v} < 0 \quad \text{and} \quad T'(x)\Big|_{x=z} > 0 .$$

Suppose the first condition holds. Then

$$a\left(\frac{w}{D(0) - v}\right)^{1-a} < 1 - e^{-rI} ,$$

or

$$I < \frac{1}{r} \ln\left(\frac{1}{e^{-rF(v+z)} - a\left(\frac{w}{z}\right)^{1-a}}\right).$$

Therefore sufficient conditions for relative minima for $x \varepsilon (D(0) - v,z]$ are that

$$\text{(9a)} \quad \frac{1}{r} \ln\left(\frac{1}{1 - a\left(\frac{w}{D(0) - v}\right)^{1-a}}\right) < I < \frac{1}{r} \ln\left(\frac{1}{e^{-rF(v+z)} - a\left(\frac{w}{z}\right)^{1-a}}\right).$$

(Note that whenever (9a) holds, this implies that (9) holds.)

<u>Remark: $T'(x)$ is continuous at $x = z$</u> .

$$T'(x)\Big|_{x=z-0} = akz^{a-1} + \frac{p}{r}\left(e^{-rF(v+z)} - e^{-rI}\right)$$

$$T'(x)\Big|_{x=z+0} = akz^{a-1}e^{-rt^*(x)}\Big|_{x=z+0} + \frac{p}{r}\left(e^{-rF(v+z)} - e^{-rI}\right).$$

But

$$t^*(x)\Big|_{x=z+0} = F(v + w^{1-a}z^a)$$

$$= F(v + w^{1-a}w^{a-1}(D(0) - v)) = F(D(0)) = 0 ,$$

hence

$$T'(x)\Big|_{x=z-0} = T'(x)\Big|_{x=z+0} .$$

<u>Case III(c): Compare $T(\hat{x})$ with $T(0)$ for relative minima for $\hat{x}\ \varepsilon\ (D(0) - v,z]$</u>

Let $\hat{x}$ be the solution to

$$a\left(\frac{w}{x}\right)^{1-a} = \left(e^{-rF(v+x)} - e^{-rI}\right)$$

which yields the smallest value of $T(x)$ over the interval $(w,z]$. Denote:

$$T(\hat{x}) = T(x)\Big|_{x=\hat{x}} \quad \text{and} \quad T(0) = T(x)\Big|_{x=0} ,$$

and

$$\Delta = T(\hat{x}) - T(0) .$$

Then

$$\Delta = k\hat{x}^a + p \int_{F(v+\hat{x})}^{I} (D(y) - v - \hat{x})e^{-ry}dy - p \int_{0}^{I} (D(y) - v)e^{-ry}dy ,$$

or

$$\Delta = k\hat{x}^a - \frac{p}{r}\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) - p \int_{0}^{F(v+\hat{x})} (D(y) - v)e^{-ry}dy .$$

This can be simplified by noting that $\hat{x}$ satisfies

$$ak\hat{x}^a = \frac{p}{r}\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) .$$

Then,

$$\Delta = (1 - a)k\hat{x}^a - p \int_{0}^{F(v+\hat{x})} (D(y) - v)e^{-ry}dy .$$

The integrand of the last term is bounded as follows:

$$0 \leq y \leq F(v + \hat{x}) ,$$

$$D(0) \leq D(y) \leq v + \hat{x} ,$$

$$D(0) - v \leq D(y) - v \leq \hat{x} .$$

Substituting $D(0) - v$, the lower bound, we can write a bound on Δ as follows:

$$\Delta \leqq (1 - a)k\hat{x}^a - \frac{p}{r}(D(0) - v)(1 - e^{-rF(v+\hat{x})}) .$$

The right side of this inequality is negative whenever

$$(1 - a)w^{1-a}\hat{x}^a < (D(0) - v)(1 - e^{-rF(v+\hat{x})}) ,$$

or

$$(1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})}) , \tag{10a}$$

because

$$(D(0) - v) = z^a w^{1-a} .$$

Hence:

$$(1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})}) \Rightarrow T(\hat{x}) < T(0) .$$

Substituting $\hat{x}$, the upper bound, we have:

$$\Delta \geqq k\hat{x}^a - \frac{p}{r}\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) - p\hat{x}\int_0^{F(v+\hat{x})} e^{-ry}dy ,$$

or

$$\Delta \geqq k\hat{x}^a - \frac{p}{r}\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) - \frac{p}{r}\hat{x}(1 - e^{-rF(v+\hat{x})}) ,$$

or

$$\Delta \geqq k\hat{x}^a\left[1 - \left(\frac{\hat{x}}{w}\right)^{1-a}(1 - e^{-rI})\right] .$$

The right side of this inequality is positive whenever:

$$1 > \left(\frac{\hat{x}}{w}\right)^{1-a}(1 - e^{-rI}) ,$$

or

$$e^{rI} < \left[\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}}\right] ,$$

or

$$I \leqq \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}}\right] .$$

Hence:

$$\text{(10b)} \qquad I \leqq \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}}\right] \Rightarrow T(\hat{x}) \geqq T(0) .$$

If neither inequality (10a) nor (10b) holds then a direct comparison of $T(\hat{x})$ with $T(0)$ must be made.

<u>Case III(d): $x \in (z, D(I) - v]$; $t^*(x) = F(v + w^{1-a}x^a) < F(v + x)$</u> .

$$T(x) = kx^a e^{-rt^*(x)} + p \int_{t^*(x)}^{I} \max\,[D(y) - v - x, 0]e^{-ry}dy$$

$$+ p \int_{0}^{t^*(x)} \max\,[D(y) - v, 0]e^{-ry}dy .$$

$$T(x) = kx^a e^{-rt^*(x)} + p\int_0^{t^*(x)} (D(y) - v)e^{-ry}dy$$

$$+ p\int_{F(v+x)}^{I} (D(y) - v - x)e^{-ry}dy$$

$$T'(x) = akx^{a-1}e^{-rt^*(x)} - \frac{p}{r}(e^{-rF(v+x)} - e^{-rI}) .$$

This is the same expression as Case II for the range

$$w < x \leqq D(I) - v \;,\; t^*(x) = F(v + w^{1-a}x^a) .$$

Note that $T'(x)$ is continuous at $x = z$ because $F(v + w^{1-a}z^a) = F(D(0)) = 0$.

<u>Case III(d): Conditions for $T'(x) \geqq 0$ for $x \in (z,D(I) - v]$</u>

Note:

$$e^{-rF(v+z)} > e^{-rt^*(x)} \geqq e^{-rF(v+w^{1-a}(D(I)-v)^a)} .$$

Also:

$$kz^{a-1} > kx^{a-1} \geqq k(D(I) - v)^{a-1} .$$

Therefore we may write a bound on the derivative as follows.

$$T'(x) \geqq ak(D(I) - v)^{a-1}e^{-rF(v+w^{1-a}(D(I)-v)^a)} + \frac{p}{r}(e^{-rI} - e^{-rF(v+z)})$$

which is nonnegative if

$$\text{(11)} \qquad a\left(\frac{w}{D(I) - v}\right)^{1-a} \geqq \frac{(e^{-rF(v+z)} - e^{-rI})}{e^{-rF(v+w^{1-a}(D(I)-v)^a)}} .$$

Hence, if (11) holds, $T(x)$ is nondecreasing throughout the interval $(z,D(I) - v]$. This is the same as we found for Case II, (see (7)), except that $F(v + z)$ has replaced $F(v + w)$. When this inequality is satisfied, relative minima do not exist in the interval $x \in (z,D(I) - v]$, although there may exist relative minima in the interval $x \in (w,z]$. Further, $T(x)$ cannot be strictly decreasing in the interval $(z,D(I) - v]$ because

$$T'(x)\Big|_{x=D(I)-v} = ak(D(I) - v)^{a-1}e^{-rF(v+w^{1-a}(D(I)-v)^{a})} > 0 .$$

If (11) does not hold, there may be relative minima for $x \in (z,D(I) - v)$.

A sufficient condition for the existence of relative minima for $x \in (z,D(I) - v)$ is

$$T'(x)\Big|_{x=z} = akz^{a-1} - \frac{p}{r}(e^{-rF(v+z)} - e^{-rI}) < 0 ,$$

or

$$a\left(\frac{w}{z}\right)^{1-a} < (e^{-rF(v+z)} - e^{-rI}) ,$$

or

(11a) $$I > \frac{1}{r}\,\ell n\left(\frac{1}{e^{-rF(v+z)} - a\left(\frac{w}{z}\right)^{1-a}}\right) .$$

(Note that whenever (8) holds, $T'(x) < 0$ for all $x \in (w,z]$, which implies that (11a) holds.) Let x_o be the solution to

$$ae^{-rF(v+w^{1-a}x^{a})}\left(\frac{w}{x}\right)^{1-a} = (e^{-rF(v+x)} - e^{-rI})$$

which yields the smallest value of $T(x)$ over the interval $(z,D(I) - v]$.

Denote:

$$T(x_o) = T(x)\Big|_{x=x_o}$$

$$T(0) = T(x)\Big|_{x=0} .$$

However, x_o is not the optimal value unless $T(x_o) < T(\hat{x}) \leq T(0)$.

<u>Case III(d): Compare $T(x_o)$ with $T(0)$ for $x \in (z, D(I) - v]$</u> .

For this case,

$$T(x_o) = kx_o^a e^{-rF\left(v+w^{1-a}x_o^a\right)} + p\int_0^{F\left(v+w^{1-a}x_o^a\right)} (D(y) - v)e^{-ry}dy$$

$$+ p\int_{F(v+x_o)}^{I} (D(y) - v - x_o)e^{-ry}dy ,$$

and

$$T(0) = p\int_0^I (D(y) - v)e^{-ry}dy .$$

Hence:

$$T(x_o) = T(0) + kx_o^a e^{-rF\left(v+w^{1-a}x_o^a\right)} - px_o\int_{F(v+x_o)}^{I} e^{-ry}dy$$

$$- p\int_{F\left(v+w^{1-a}x_o^a\right)}^{F(v+x_o)} (D(y) - v)e^{-ry}dy .$$

Except for the range on "x" , this is the same result as we found for Case II where $x \in (w, D(I) - v]$; whereas, for this case $x \in (z, D(I) - v]$. Accordingly we can write:

$$T(x_o) < T(0) \quad \text{if} \quad I > F(v + x_o) + \frac{1}{r}\left(\ell n \left(\frac{x_o}{w}\right)^{1-a} - \ell n \left(\left(\frac{x_o}{w}\right)^{1-a} - 1\right)\right)$$

$$T(x_o) \geqq T(0) \quad \text{if} \quad I \leqq F\left(v + w^{1-a} x_o^a\right) + \frac{1}{r}\left(\ell n \left(\frac{x_o}{w}\right)^{1-a} - \ell n \left(\left(\frac{x_o}{w}\right)^{1-a} - 1\right)\right) .$$

But if neither holds, i.e., if

$$F\left(v + w^{1-a} x_o^a\right) + \frac{1}{r}\,\ell n \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] < I \leqq F(v + x_o) + \frac{1}{r}\,\ell n \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] ,$$

then the role of x_o is uncertain and $T(x_o)$ must be compared directly with $T(0)$.

<u>Remark</u>:

It is impossible for $T'(x) < 0$ for all $x \in (w, z]$ and $T'(x) \geqq 0$ for all $x \in (z, D(I) - v]$ because $T'(x)$ is continuous at $x = z$. Furthermore, inequalities (8) and (11) cannot hold simultaneously because if

$$\text{(8)} \qquad a < e^{-rF(v+z)} - e^{-rI}$$

and

$$\text{(11)} \qquad a\left(\frac{w}{D(I) - v}\right)^{1-a} e^{-rF(v+w^{1-a}(D(I)-v)^a)} \geqq (e^{-rF(v+z)} - e^{-rI})$$

hold, this implies that

$$a\left(\frac{w}{D(I) - v}\right)^{1-a} e^{-rF(v+w^{1-a}(D(I)-v)^{a})} > a$$

which is impossible.

Case III Summary:

Recall inequality (8), (8a), (9), (9a), (11) and the implications when each holds:

$$\left\{(8): \quad I > \frac{1}{r} \ell n \left(\frac{1}{e^{-rF(v+z)} - a}\right)\right\} \Rightarrow T'(x) < 0 \qquad \forall\ x \in (w,z] ,$$

$$\left\{(8a): \quad I \leqq \frac{1}{r} \ell n \left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right)\right\} \Rightarrow T'(x) \geqq 0 \qquad \forall\ x \in (w,z] ,$$

$$\left\{(9): \quad \frac{1}{r} \ell n \left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right) < I \leqq \frac{1}{r} \ell n \left(\frac{1}{e^{-rF(v+z)} - a}\right)\right\} \Rightarrow \text{the sign of}$$

$T'(x)$ is uncertain $\forall\ x \in (w,z]$,

$$\left\{(9a): \quad \frac{1}{r} \ell n \left(\frac{1}{1 - a\left(\frac{w}{D(0) - v}\right)^{1-a}}\right) < I < \frac{1}{r} \ell n \left(\frac{1}{e^{-rF(v+z)} - a\left(\frac{w}{z}\right)^{1-a}}\right)\right\} \Rightarrow$$

$$T'(x)\Big|_{x=D(0)-v} < 0 \quad \text{and} \quad T'(x)\Big|_{x=z} > 0 ,$$

$$\left\{(11): \quad a\left(\frac{w}{D(I) - v}\right)^{1-a} \geqq \left(\frac{e^{-rF(v+z)} - e^{-rI}}{e^{-rF(v+w^{1-a}(D(I)-v)^{a})}}\right)\right\} \Rightarrow T'(x) \geqq 0$$

$\forall x \in (z,D(I) - v]$.

We conclude that:

(a) If both (11) and (8a) hold, there are no relative minima anywhere in the interval $(0,D(I) - v]$ and hence $x^{*} = 0$.

(b) If (11) and (9) hold, there may exist relative minima in $(D(o)-v,\ z]$, and <u>only</u> in this interval. And, for $\hat{x}$ yielding the smallest value of $T(x)$ in this interval:

$$x^* = \hat{x} \text{ if } (1-a)\left(\frac{\hat{x}}{z}\right)^a < \left(1 - e^{-rF(v+\hat{x})}\right),$$

$$x^* = 0 \text{ if } I \leq \frac{1}{r}\ \ell n\left[\frac{1}{1-\left(\frac{\hat{x}}{w}\right)^{1-a}}\right].$$

and $T(\hat{x})$ must be compared directly with $T(0)$ if neither of the above tests is satisfied.

(c) If (9) holds but (11) does not, there may be relative minima in either or both intervals:

$(D(0)-v,\ z]$, $(z,\ D(I)-v)$. And, if they exist in both intervals, $T(0)$ must be compared with $T(x_o)$ and $T(\hat{x})$.

(d) If (11) and (9a) hold, then, there exist relative minima in $(D(0) - v, z]$ and *only* in this interval. And, for $\hat{x}$ yielding the smallest value of $T(x)$ in this interval:

$$x^* = \hat{x} \quad \text{if} \quad (1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})}) ,$$

$$x^* = 0 \quad \text{if} \quad I \leqq \frac{1}{r} \ln \left(\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{1-a}}\right) ,$$

and $T(\hat{x})$ must be compared directly with $T(0)$ if neither of the above tests is satisfied.

(e) Relative minima exist exclusively in $(z, D(I) - v)$ only if $T'(x) < 0$ for some $x \in (z, D(I) - v)$ and either $T'(x) < 0$ for all $x \in (D(0) - v, z]$ or $T'(x) \geqq 0$ for all $x \in (D(0) - v, z]$, which requires that (11) does not hold and (8) or (8a) holds. If these conditions hold,

$$x^* = x_o \quad \text{if} \quad I > F(v + x_o) + \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] ,$$

and

$$x^* = 0 \quad \text{if} \quad I \leqq F\left(v + w^{1-a} x_o^a\right) + \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] .$$

But if

$$F\left(v + w^{1-a} x_o^a\right) + \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] < I \leqq F(v + x_o) + \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{w}{x_o}\right)^{1-a}}\right] ,$$

the role of x_o is uncertain and $T(x_o)$ must be compared directly with $T(0)$.

(f) Otherwise, (11) does not hold but (9a) does hold and, relative minima may exist in either or both intervals, $(D(0) - v,z]$, $(z,D(I) - v)$ and if they exist in both, $T(0)$ must be compared with $T(x_o)$ and $T(\hat{x})$, by using the tests above or by direct comparison.

Case IV: $w < D(0) - v$ and $z \geqq D(I) - v$

(See Figure 5). This case is similar to Case III for the range $x \in (w,z]$. For Case IV we found that

$$t^*(x) = \begin{cases} I & \text{for } x \in [0,w] \\ 0 & \text{for } x \in (w,D(I) - v] \end{cases} .$$

Accordingly, most of the required analysis has been done and we may summarize the conclusions from Case III which apply to Case IV as follows:

a. $T(x)$ is strictly concave, increasing for all x in the interval $[0,w]$.

b. $T(x)$ is strictly concave for all $x \in (0,D(0) - v]$ and $T'(x)$ is continuous at $x = D(0) - v$.

c. Since there are no relative minima for $x \in (w,D(0) - v]$, we need only consider the range $x \in (D(0) - v , D(I) - v]$, for which $t^*(x) = 0$.

Case IV(a): $x \in (D(0) - v,D(I) - v]$; $t^*(x) = 0$

We can write,

$$T(x) = kx^a + p \int_{F(v+x)}^{I} (D(y) - v - x)e^{-ry}dy \ ,$$

and

$$T'(x) = akx^{a-1} - \frac{p}{r} (e^{-rF(v+x)} - e^{-rI}) \ .$$

Now at the end of the interval,

$$T'(x)\Big|_{x=D(I)-v} = ak(D(I) - v)^{a-1} > 0$$

as in Case III, and $T(x)$ cannot be decreasing throughout the entire interval $(D(0) - v, D(I) - v]$.

The existence of relative minima in this interval requires that $T'(x) < 0$ for some $x \in (D(0) - v, D(I) - v]$. As before, we look for conditions which rule out such a possibility, namely, conditions for which $T'(x) \geqq 0$ for all $x \in (D(0) - v, D(I) - v]$. As before we note that for $x \in (D(0) - v, D(I) - v]$:

$$1 > e^{-rF(v+x)} \geqq e^{-rI} \ ,$$

and

$$k(D(0) - v)^{a-1} > kx^{a-1} \geqq k(D(I) - v)^{a-1} \ .$$

Hence a lower bound on the derivative may be written as follows:

$$T'(x) \geqq ak(D(I) - v)^{a-1} - \frac{p}{r} (1 - e^{-rI}) \ .$$

The right side of this inequality is positive or zero whenever

$$a(D(I) - v)^{a-1} w^{1-a} \geq (1 - e^{-rI}) \, ,$$

or as a condition on I :

$$I \leq \frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{D(I) - v}\right)^{1-a}} \right) . \tag{12}$$

Hence we conclude that

$$\left\{ I \leq \frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{D(I) - v}\right)^{1-a}} \right) \right\} \Rightarrow T'(x) \geq 0 \quad \forall\, x \in (D(0) - v, D(I) - v] \, .$$

Suppose (12) holds. Then, since

$$(D(0) - v)^{a-1} \geq (D(I) - v)^{a-1} \, ,$$

we can write

$$T'(x)\Big|_{x=D(0)-v} = ak(D(0) - v)^{a-1} - \frac{p}{r}(1 - e^{-rI})$$

which is strictly positive. But $T(x)$ is concave for all $x \in (w, D(0) - v]$; hence, whenever (12) holds, $T(x)$ is strictly increasing for all $x \in (w, D(0) - v]$. Since $T(x)$ is also concave increasing for all $x \in (0, w]$, it follows that whenever (12) holds, $T(x)$ is nondecreasing over the entire interval $[0, D(I) - v]$. We conclude that:

$$\left\{ I \leq \frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{D(I) - v}\right)^{1-a}} \right) \right\} \Rightarrow x^* = 0 \, ,$$

and when this inequality does not hold, there may exist relative minima in the open interval $x \in (D(0) - v, D(I) - v)$. A sufficient condition for the existence of relative minima in $(D(0) - v, D(I) - v]$ is $T'(x)\Big|_{x=D(0)-v} < 0$

or

$$a\left(\frac{D(0) - v}{w}\right)^{a-1} < 1 - e^{-rI}$$

or

(12a) $$I > \frac{1}{r} \ln\left(\frac{1}{1 - a\left(\frac{w}{D(0) - v}\right)^{1-a}}\right).$$

Let $\hat{x}$ be the solution to

$$a\left(\frac{w}{x}\right)^{1-a} = (e^{-rF(v+x)} - e^{-rI})$$

which yields the smallest value of $T(x)$ over the open interval $(D(0) - v, D(I) - v)$. But $\hat{x}$ is not the optimal value unless $T(\hat{x}) < T(0)$ where

$$T(\hat{x}) = T(x)\Big|_{x=\hat{x}}$$

$$T(0) = T(x)\Big|_{x=0}.$$

Denote:

$$\Delta = T(\hat{x}) - T(0)$$

$$\Delta = k\hat{x}^a - \frac{p}{r}\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) - p\int_0^{F(v+\hat{x})} (D(y) - v)e^{-ry}dy .$$

The integrand is bounded as follows:

$$0 \leq y \leq F(v + \hat{x}) \ ,$$

$$D(0) \leq D(y) \leq v + \hat{x} \ ,$$

$$D(0) - v \leq D(y) - v \leq \hat{x} \ .$$

Substituting the lower bound, gives:

$$\Delta \leq k\hat{x}^a - \frac{p}{r}\,\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) + \frac{p}{r}\,(D(0) - v)(e^{-rF(v+\hat{x})} - 1) \ .$$

As in Case III, the right side of this inequality is negative whenever

$$(1 - a)k\hat{x}^a < \frac{p}{r}\,(D(0) - v)(1 - e^{-rF(v+\hat{x})}) \ ,$$

or

$$\text{(10a)} \qquad (1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})}) \ .$$

Hence, as in Case III:

$$(1 - a)w^{1-a}\hat{x}^a < (D(0) - v)(1 - e^{-rF(v+\hat{x})}) \Rightarrow T(\hat{x}) < T(0) \ .$$

Substituting $\hat{x}$, the upper bound, we have:

$$\Delta \geq k\hat{x}^a - \frac{p}{r}\,\hat{x}(e^{-rF(v+\hat{x})} - e^{-rI}) - \frac{p}{r}\,\hat{x}(1 - e^{-rF(v+\hat{x})}) \ ,$$

$$\Delta \geq k\hat{x}^a\left[1 - \left(\frac{\hat{x}}{w}\right)^{1-a}(1 - e^{-rI})\right] .$$

Again as in Case III, the right side of this inequality is positive whenever:

$$e^{rI} < \left[\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}}\right] ,$$

or

(10b) $$I < \frac{1}{r} \ln \left(\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}} \right) .$$

hence:

$$\left\{ I \leq \frac{1}{r} \ln \left[\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}} \right] \right\} \Rightarrow T(\hat{x}) \geq T(0) ,$$

the same as we found for Case III. If neither (10a) nor (10b) holds, a direct comparison of $T(\hat{x})$ with $T(0)$ must be made.

Case IV: Summary

(a) $x \in [0,w]$.

$t^*(x) = I$ and $T(x)$ is concave increasing for all $x \in [0,w]$.

(b) $x \in (w,D(I) - v]$.

$$I \leq \frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{D(I) - v}\right)^{1-a}} \right) \Rightarrow T'(x) \geq 0 \quad \forall\, x \in (w,D(I) - v]$$

and, hence, $x^* = 0$.

But when this inequality does not hold, there may exist relative minima for $x \in (D(0) - v,D(I) - v)$, and $\hat{x}$ yields the smallest value of $T(x)$ in this interval. But $\hat{x}$ may not be the optimal value, unless $T(\hat{x}) < T(0)$. And:

$$x^* = \hat{x} \quad \text{if} \quad (1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})})$$

$$x^* = 0 \quad \text{if} \quad I \leq \frac{1}{r} \ln \left(\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{a-1}} \right)$$

and if neither of these inequalities hold, then $T(\hat{x})$ must be compared directly with $T(0)$.

SUMMARY OF ALL CASES FOR OPTIMAL x

Case I: $w \geq (D(I) - v)$.

$$t^*(x) = I \ , \ x^* = 0 \ .$$

Case II: $D(0) - v < w < D(I) - v$.

$$\left\{ \begin{array}{c} x^* = 0 \\ t^*(x) = I \end{array} \right\} \text{ if } \ I \leq -\frac{1}{r} \ln \left[e^{-rF(v+w)} - \left(\frac{w}{D(I) - v} \right)^{1-a} a e^{-rF\left(v + w^{1-a}(D(I)-v)^a\right)} \right] \ .$$

Let $T(x_o)$ be the smallest value of $T(x)$ where x_o satisfies

$$akx^{a-1} e^{-rF(v+w^{1-a}x^a)} = \frac{p}{r} \left(e^{-rF(v+x)} - e^{-rI} \right) \ .$$

Let:

$$U(x_o,v,r,w,a) = F(v + x_o) + \frac{1}{r}\left[\ln \left(\frac{x_o}{w} \right)^{1-a} - \ln \left(\left(\frac{x_o}{w} \right)^{1-a} - 1 \right) \right]$$

$$L(x_o,v,r,w,a) = F\left(v + w^{1-a}x_o^a\right) + \frac{1}{r}\left[\ln \left(\frac{x_o}{w} \right)^{1-a} - \ln \left(\left(\frac{x_o}{w} \right)^{1-a} - 1 \right) \right] \ .$$

Then:

$$\left\{ \begin{array}{c} x^* = x_o \\ t^*(x_o) = F\left(v + w^{1-a}x_o^a\right) \end{array} \right\} \text{ if } \ I > U(x_o,v,r,w,a)$$

$$\left\{ \begin{array}{c} x^* = 0 \\ t^*(x) = I \end{array} \right\} \text{ if } \ I \leq L(x_o,v,r,w,a) \ .$$

and if $L(x_o,v,r,w,a) < I \leq U(x_o,v,r,w,a)$, a direct comparison is required.

Case III: $w < D(0) - v$ and $z < D(I) - v$.

Recall:

(8) $$I > \frac{1}{r} \ln \left(\frac{1}{e^{-rF(v+z)} - a}\right)$$

(8a) $$I \leq \frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right)$$

(9) $$\frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{z}\right)^{1-a}}\right) < I \leq \frac{1}{r} \ln \left(\frac{1}{e^{-rF(v+z)} - a}\right)$$

(9a) $$\frac{1}{r} \ln \left(\frac{1}{1 - a\left(\frac{w}{D(0) - v}\right)^{1-a}}\right) < I < \frac{1}{r} \ln \left(\frac{1}{e^{-rF(v+z)} - a\left(\frac{w}{z}\right)^{1-a}}\right)$$

(11) $$a\left(\frac{w}{D(I) - v}\right)^{1-a} \geq \left(\frac{e^{-rF(v+z)} - e^{-rI}}{e^{-rF(v+w^{1-a}(D(I)-v)^{a})}}\right).$$ Note: (9a) => (9).

If (11) and (8a) hold, then $x^* = 0$ because $T(x)$ is nondecreasing throughout the entire interval.

If (11) and (9) hold there are no relative minima in $(z, D(I) - v]$ but there may be relative minima in $(D(0) - v, z]$; and, if there are:

$$\left\{\begin{matrix} x^* = \hat{x} \\ t^*(\hat{x}) = 0 \end{matrix}\right\} \text{ if } (1 - a)\left(\frac{\hat{x}}{z}\right)^a < (1 - e^{-rF(v+\hat{x})}) ,$$

$$\{x^* = 0\} \text{ if } I \leq \frac{1}{r} \ln \left(\frac{1}{1 - \left(\frac{\hat{x}}{w}\right)^{1-a}}\right) ,$$

and $T(\hat{x})$ must be compared with $T(0)$ if neither of these tests apply, in which case,

$$\left\{\begin{array}{c} x^* = \dot{x} \\ t^*(\dot{x}) = 0 \end{array}\right\} \text{ if } T(\dot{x}) < T(0) ,$$

$$\{x^* = 0\} \text{ if } T(\dot{x}) \geq T(0) .$$

If (11) does not hold but (8) or (8a) holds, there are no relative minima in $(w,z]$ but there may be relative minima in $(z,D(I)-v)$; and if there are:

$$\left\{\begin{array}{c} x^* = x_o \\ t^*(x_o) = F\left(v + w^{1-a}x_o^a\right) \end{array}\right\} \text{ if } I > U(x_o,v,r,w,a) ,$$

$$\{x^* = 0\} \text{ if } I < L(x_o,v,r,w,a) ,$$

but if $L(x_o,v,r,w,a) < I \leq U(x_o,v,r,w,a)$, a direct comparison must be made and

$$\left\{\begin{array}{c} x^* = x_o \\ t^*(x_o) = F\left(v + w^{1-a}x_o^a\right) \end{array}\right\} \text{ if } T(x_o) < T(0) ,$$

$$\{x^* = 0\} \text{ if } T(x_o) \geq T(0) .$$

If (9) holds but not (11) there may be relative minima in both intervals, $(D(0) - v,z]$ and $(z,D(I) - v]$ and these must be compared with $T(0)$. And,

$$\left\{\begin{array}{c} x^* = x_o \\ t^*(x_o) = F\left(v + w^{1-a}x_o^a\right) \end{array}\right\} \text{ if } \left\{\begin{array}{l} T(x_o) < T(0) \text{ and} \\ T(x_o) < T(\hat{x}) \end{array}\right\} ,$$

$$\left\{\begin{array}{l} x^{*} = \hat{x} \\ t^{*}(\hat{x}) = 0 \end{array}\right\} \text{ if } \left\{\begin{array}{l} T(\hat{x}) < T(0) \text{ and} \\ T(\hat{x}) < T(\lambda) \end{array}\right\}$$

and

$$(x = 0) \qquad \text{if} \quad \left\{\begin{array}{l} T(0) \leq T(x_0) \text{ and} \\ T(0) \leq T(\hat{x}) \end{array}\right\} .$$

Case IV: $W < D(0)-v$ and $z \geq D(I)-v$.

Recall:

$$I \leq \frac{1}{r} \cdot \ln\left(\frac{1}{1 - a\left(\frac{v}{(D(I)-v}\right)^{1-a}}\right) \tag{12}$$

If (12) holds, there are no relative minima in the interval $(w, D(I)-v]$ and $x^{*} = 0$. Otherwise, there may be relative minima in the open interval, $(w, D(I)-v)$. But, $\hat{x}$, the value of x, for which $T(x)$ is a minimum in this interval, is not the optimal value unless $T(\hat{x}) < T(0)$.

Then:

$$x^{*} = \hat{x} \text{ if } (1-a)\left(\frac{\hat{x}}{z}\right)^{a} < (1 - e^{-rF(v + \hat{x})})$$

$$x^{*} = 0 \text{ if } I \leq \frac{1}{r} \ln\left[\frac{1}{1-\left(\frac{\hat{x}}{w}\right)^{a-1}}\right]$$

and if neither of these tests are met, a direct comparison of $T(\hat{x})$ with $T(0)$ is required.

<u>Remarks on uniqueness of relative minima $\dot{x}$ and x_o</u>

In our analysis of Cases II, III and IV we found that whenever relative minima exist on the interior of $x \varepsilon (w,D(I) - v)$, they are for values of x which satisfy $T'(x) = 0$, or:

$$T'(x) = akx^{a-1}e^{-rt^*(x)} - \frac{p}{r}(e^{-rF(v+x)} - e^{-rI}) = 0$$

Accordingly, denote $S(x)$ for $x \varepsilon (w,D(I) - v]$ as follows:

$$S(x) = akx^{a-1}\, e^{-rt^*(x)} - \frac{p}{r}\,(e^{-rF(v+x)} - e^{-rI}),$$

where:

$$t^*(x) = \left\{\begin{array}{lll} F(v + w^{1-a}x^a) & \text{for } x \varepsilon (w,D(I) - v] & \text{for Case II,} \\ F(v + w^{1-a}x^a) & \text{for } x \varepsilon (z,D(I) - v] & \text{for Case III,} \\ 0 & \text{for } x \varepsilon (D(0) - v,z] & \text{for Case III,} \\ 0 & \text{for } x \varepsilon (D(0) - v,I] & \text{for Case IV} \end{array}\right\}.$$

Denote:

$$H(x) = akx^{a-1}e^{-rt^*(x)},$$

$$G(x) = \frac{p}{r}\,(e^{-rF(v+x)} - e^{-rI}) .$$

Then:

$$S(x) = H(x) - G(x) \ ,$$

$$S'(x) = H'(x) - G'(x) \ ,$$

and

$$S''(x) = H''(x) - G''(x) \ .$$

Then:

$$H'(x) = a(a-1)kx^{a-2}e^{-rt^*(x)} - rakx^{a-1}e^{-rt^*(x)}\left(\frac{d}{dx}\ t^*(x)\right) \ .$$

And

$$H''(x) = a(a-1)(a-2)kx^{a-3}e^{-rt^*(x)} + (1-a)rakx^{a-2}e^{-rt^*(x)}\left[\frac{d}{dx}\ t^*(x)\right]$$

$$+\ r^2akx^{a-1}e^{-rt^*(x)}\left[\frac{d}{dx}\ t^*(x)\right]^2 - rakx^{a-1}e^{-rt^*(x)}\left[\frac{d^2}{dx^2}\ t^*(x)\right] \ ,$$

which is positive whenever $\frac{d^2}{dx^2}\ t^*(x)$ is negative or zero because $\frac{d}{dx}\ t^*(x)$ is nonnegative. ($H''(x) > 0$ for $t^*(x) = 0$) . Now:

$$G'(x) = pe^{-rF(v+x)}\ \frac{d}{dx}\ F(v+x) > 0 \ .$$

And:

$$G''(x) = pe^{-rF(v+x)}\left(-r\left[\frac{d}{dx}\ F(v+x)\right]^2 + \frac{d^2}{dx^2}\ F(v+x)\right) \ ,$$

which is nonpositive whenever $\frac{d^2}{dx^2}\ F(v+x)$ is nonpositive, which in turn implies $-G''(x)$ is positive.

Hence $S''(x)$ is positive and $S(x)$ is strictly convex whenever both $\frac{d^2}{dx^2} F(v + x)$ and $\frac{d^2}{dx^2} t^*(x)$ are nonpositive. One class of demand functions for which these conditions hold is $D(y)$ convex because $F(x)$ is concave. For such a class, there are at *most* two zeros of $S(x)$.

For example, suppose $v = 0$, $D(y) - e^y$ and $w = 1$. Then Case II applies and $t^*(x) = F(v + w^{1-a}x^a)$. Restrict $x \in (w,D(I)] \Rightarrow 1 < x < e^I$. Then

$$\ell n\ D(y) = y$$

$$F(D(y)) = F(e^y) = \ell n\ D(y) = y\ ,$$

$$F(v + x) = \ell n\ (v + x) = \ell n\ x\ ,$$

$$\frac{d}{dx} F(v + x) = \frac{1}{x}\ ,$$

$$t^*(x) = F(x^a) = a\ \ell n\ x\ ,$$

$$\frac{d}{dx} t^*(x) = \frac{a}{x}\ ,\ \frac{d^2}{dx^2} t^*(x) = -ax^{-2}$$

$$\frac{d^2}{dx^2} F(v + x) = -x^{-2} < 0\ .$$

Further:

$$H''(x) = ka(a - 1)(a - 2)x^{a-3}e^{-rt^*(x)} + (1 - a)rakx^{a-2}e^{-rt^*(x)}\left(\frac{a}{x}\right)$$
$$+ r^2ax^{a-1}e^{-rt^*(x)}\left(\frac{a}{x}\right)^2 + rax^{a-1}e^{-rt^*(x)}(ax^{-2})\ .$$

Hence $H''(x) > 0$. Now:

$$G''(x) = pe^{-r\ell nx}\left(-r\left(\frac{1}{x^2}\right) - x^{-2}\right) < 0\ .\ .$$

Hence $-G''(x) > 0$, $S''(x) = H''(x) - G''(x) > 0$ and $S(x)$ is convex with at most two zeros.

For Case II and IV when $t^*(x) = 0$ for all $x \in (w,z]$:

$$H(x)\Big|_{x\epsilon(w,z)} = akx^{a-1} > 0 ,$$

and $H(x)$ is convex, decreasing. Hence these remarks apply to all cases, except Case I, for $x \in (w,D(1) - v]$.

An Algorithm for Finding an Optimal Expansion Policy

1. Define whether Case I, II, III, or IV applies as follows: Compute $w - (D(I)-v)$. If non-negative, Case I applies. Otherwise, compute $w - (D(0)-v)$. If non-negative, Case II applies. Otherwise, compute $D(I)-v-z$. If non-negative, Case III applies; otherwise Case IV applies. If Case I applies, $x^* = 0$. For Cases II, III, IV, go to stops 2,3,4 respectively.

2. (Case II). Compute:

$$a\left(\frac{w}{D(I)-v}\right)^{1-a} e^{-rF(v+w^{1-a}(D(I)-v)^{a})} + e^{-rI} - e^{-rF(v+w)} .$$

If non-negative $x^* = 0$ because $T(x)$ has no relative minima exist in the interior of $(0,D(I)-v]$.

Otherwise go to step 5.

3. Case III. Compute Δ_1 and Δ_2 where:

$$\Delta_1 = a\left(\frac{w}{D(I)-v}\right)^{1-a} - \left(\frac{e^{-rF(v+z)} - e^{-rI}}{e^{-rF(v+w^{1-a}(D(I)-v)^{a})}}\right) ,$$

and

$$\Delta_2 = a - \left(\frac{z}{w}\right)^{1-a} (1-e^{-rI})$$

Remark:

$\Delta_1 \geqq 0 \Rightarrow$ inequality (11) holds, and

$\Delta_2 \geqq 0 \Rightarrow$ inequality (9) holds $\Rightarrow$ (10) does not hold.

If $\Delta_1 \geqq 0$ and $\Delta_2 \geqq 0$, $x^* = 0$ because $T(x)$ has no relative minima in $(0, D(I)-v]$.

If $\Delta_1 < 0$ and $\Delta_2 \geqq 0$, there may be relative minime, but only in $(z, D(I)-v)$; hence go to step 5.

If $\Delta_1 \geqq 0$ and $\Delta_2 < 0$, there may be relative minima, but only in $(D(0)-v, z]$; hence, go to step 6.

If $\Delta_1 < 0$ and $\Delta_2 < 0$, there may be relative minima in both intervals $(D(0)-v, z]$ and $(z, D(IO-v)$; hence, go to step 7.

4. Case IV. Compute

$$\frac{1}{r}\,\ell n\left(\frac{1}{1 - a\left(\frac{w}{D(I)-v}\right)^{1-a}}\right) - I.$$

If non-negative, $x^* = 0$ because $T(x)$ has no local minima in the interior of $(0, D(I)-v]$. Otherwise go to step 6.

5. (Cases II and III). Compute all solutions to:

$$a\left(\frac{x}{w}\right)^{a-1}\left(e^{-rF(v+w^{1-a}x^{a})}\right) = \left(e^{-rF(v+x)} - e^{-rI}\right)$$

where: $x \in (w, D(I) - v)$ for Case II and $x \in (z, D(I)-v)$ for Case III. Newton's Method or some other appropriate method of numerical analysis may be employed. If $D(y)$ is convex, there are at most two solutions.

Denoting x_o as the "best" solution, compute

$I - L(x_o, v, r, w, a)$ where

$$L(x_o,v,r,w,a) = F(v + x_o) + \frac{1}{r} \ln\left[\frac{1}{1-\left(\frac{w}{x_o}\right)^{(1-a)}}\right] .$$

If the result is positive, $x^* = x_o$. Otherwise, compute

$I - U(x_o,v,r,w,a)$

where:

$$U(x_o,v,r,w,a) = F(v + w^{1-a}x_o^a) + \frac{1}{r} \ln\left[\frac{1}{1-\left(\frac{w}{x_o}\right)^{(1-a)}}\right] .$$

If this result is negative or zero, $x^* = 0$. Otherwise, compute $T(x_o) - T(0)$. If this result is negative, $x^* = 0$; otherwise $x^* = 0$.

6. (Cases III and IV). Compute all solutions to

$$a\left(\frac{x}{w}\right)^{1-a} = e^{-rF(v+x)} - e^{-rI} ,$$

where $x \in (D(0)-v,\ z]$ for Case III and $x \in (D(0)-v,\ D(I)-v)$ for Case IV.

If $D(y)$ is convex, there are at most two solutions. Denoting $\hat{x}$ as the best solution, compute

$$(1-a) \quad \left(\frac{\hat{x}}{z}\right)^a - (1-e^{-rF(v+\hat{x})}).$$

If the result is negative, $x^* = \hat{x}$. Otherwise, compute

$$I - \frac{1}{r} \ln\left[\frac{1}{1-\left(\frac{\hat{x}}{w}\right)^{1-a}}\right] .$$

If the result is non-negative, $x^* = 0$. Otherwise, compute $T(\hat{x}) - T(0)$. If the result is negative, $x^* = \hat{x}$, otherwise $x^* = 0$.

7. (Case III: both $\hat{x}$ and x_o may exist.) Compute all solutions to

$$a\left(\frac{x}{w}\right)^{a-1} e^{-rF(v+w^{1-a}x^a)} = (e^{-rF(v+x)} - e^{-rI}),$$

where $x \in (z, D(I)-v)$, and x_o is the best solution. Compute all solutions to

$$a\left(\frac{x}{w}\right)^{a-1} = (e^{-rF(v+x)} - e^{-rI}),$$

where $x \in (D(0)-v, z]$, and $\hat{x}$ as the best solution. Compute $T(x_o) - T(0)$, and $T(\hat{x}) - T(0)$.

Then:

$$x^* = x_o \quad \text{if} \quad \left\{\begin{array}{l} T(x_o) < T(0) \text{ and} \\ T(x_o) < T(\hat{x}) \end{array}\right\},$$

$$x^* = \hat{x} \quad \text{if} \quad \left\{\begin{array}{l} T(\hat{x}) < T(0) \text{ and} \\ T(\hat{x}) < T(x_o) \end{array}\right\},$$

$x^* = 0$ otherwise.

CHAPTER IV

AN ALGORITHM FOR SEQUENTIAL CAPACITY EXPANSIONS

Introduction

In this chapter we consider a sequence of capacity expansions over time. The algorithm of Chapter III for a single expansion is too cumbersome to use for sequential expansions. Accordingly, we present an algorithm for digital computers in which demand and capacity take on discrete values.

Our approach is to treat an n-expansion problem of $2n$ decision variables as an n-stage decision process. Two types of problems are considered. The first, Type I, does not permit a shortage of capacity after the last expansion while a Type II problem does. We show that solving a Type II problem requires that a Type I problem also be solved.

Let x_j denote the size of the jth capacity expansion, t_j the time at which it occurs and v_j the capacity before it occurs. Restrict the times of capacity expansions as follows:

$$0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq I .$$

Type I Problems

Recall that $t^*(x)$ for a single expansion, Cases I through IV, Chapter III can be summarized as follows (see Figures 2 through 5):

$$\text{(13)} \qquad t^*(x) = \begin{Bmatrix} I & \text{if} \quad x \in [0,w] \\ 0 & \text{if} \quad x \in (w,z) \quad \text{and} \quad w + v < D(0) \\ F(v + w^{1-a}x^a) & \text{otherwise} \end{Bmatrix}$$

where I is the length of the planning interval and $t \in [0,I]$.

Suppose we restrict x such that $x = D(I) - v$ and allow $(D(I) - v)$ to vary. This forces the last term of $T(x)$ to vanish, and

$$\text{(13a)} \qquad p \int_t^I \max\,(D(y) - v - x, 0)e^{-ry}dy \equiv 0 \ .$$

Let $C(\cdot)$ denote the single expansion cost function under this restriction. To avoid ambiguity, let the last expansion for a j-stage problem satisfy:

$$x_j = D(i) - v_j \ ,$$

where

$$D(i) \in [D(0), D(I)]$$

and

$$v_j \in [v_o, D(I)] \ .$$

Note, we are also defining v_j, to be a parameter, and v_o to be the initial capacity at time zero.

Remark:

Note that by our restriction on x, $t^*(x) \leqq F(v + x)$. Hence, the configuration implied by our definition of Type I problems is shown below (for $v_1 < D(0)$).

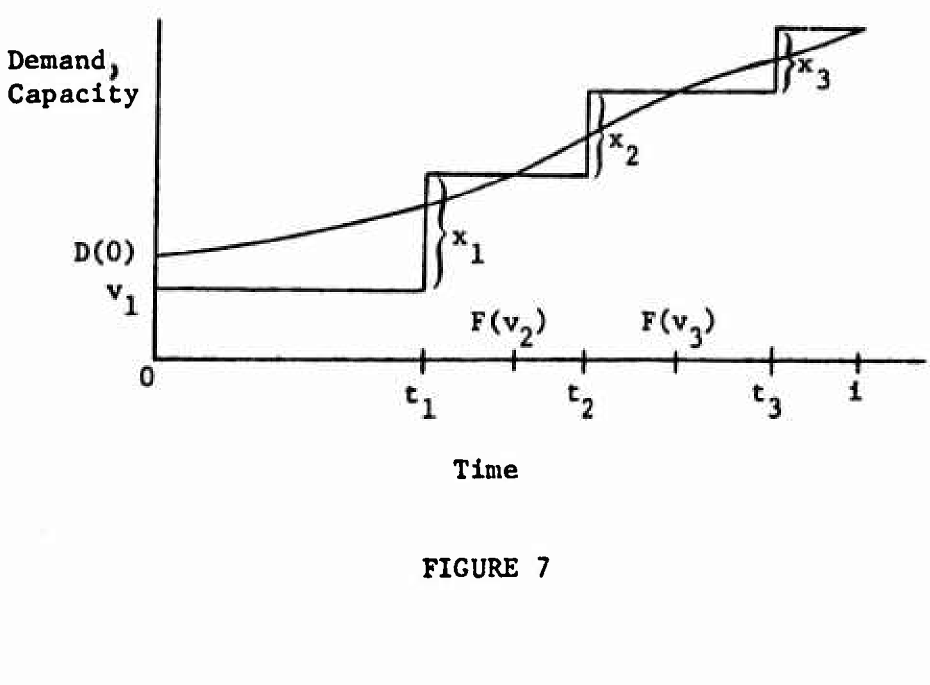

FIGURE 7

In particular, we can write without loss of generality that

$$0 \leq t_1^*(x_1) \leq F(v_1 + x_1) \leq t_2^*(x_2) \leq F(v_2 + x_2) \leq \dots \leq I .$$

This will permit us to simplify the cost functions. Note also that we can define the jth stage unambiguously as beginning at $v_j = F(v_1 + x_1 + \dots + x_{j-1})$ and ending $v_{j+1} = F(v_1 + x_1 + \dots + x_j)$.

Restricting x , so that (13a) holds, the cost function for the Type I single stage problem is

$$C_1(x_1,v_1) = kx_1^a e^{-rt_1^*(x_1)} + p \int_0^{t_1^*(x_1)} \max\,(D(y) - v_1, 0)e^{-ry}dy ,$$

where:

$$x_1 = (D(i) - v_1) ,$$

and

$$t_1^*(x_1) = \left\{\begin{array}{ll} t_2^*(x_2) & \text{if } x_1 \in [0,w] \\ 0 & \text{if } x_1 \in (w,z) \text{ and } w + v_1 < D(0) \\ F\left(v_1 + w^{1-a}x_1^a\right) & \text{otherwise} \end{array}\right\} .$$

Note that $t_2^*(x_2) \leq I$ with equality always holding for a one-stage problem.

The cost function for the first two stages of a Type I problem is given by the sum:

$$C_1(x_1,v_1) + C_2(x_2,v_2) = kx_1^a e^{-rt_1^*(x_1)} + p \int_0^{t_1^*(x_1)} \max\,(D(y) - v_1, 0)e^{-ry}dy$$

$$+ kx_2^a e^{-rt_2^*(x_2)} + p \int_{t_1^*(x_1)}^{t_2^*(x_2)} \max\,(D(y) - v_2, 0)e^{-ry}dy$$

where

$$x_1 + x_2 = D(i) - v_1 \ ,$$

$$v_2 = v_1 + x_1 = D(i) - x_2 \ ,$$

$$t_2^*(x_2) = \begin{cases} t_3^*(x_3) & \text{if } x_2 \in [0,w] \\ t_1^*(x_1) & \text{if } x_2 \in (w,z) \text{ and } w + v_2 < D(0) \\ F\left(v_2 + w^{1-a}x_2^a\right) & \text{otherwise} \end{cases} ,$$

and $t_3^*(x_3) = I$ if this be only a two-stage problem. This can be simplified to obtain symmetry of definition. Write $C_1(x_1,v_1)$ in an equivalent form:

$$C_1(x_1,v_1) = kx_1^a e^{-rt_1^*(x_1)} + p \int_0^{F(v_1)} \max\,(D(y) - v_1, 0)e^{-ry}dy$$

$$+ p \int_{F(v_1)}^{t_1^*(x_1)} \max\,(D(y) - v_1, 0)e^{-ry}dy \ .$$

Examine the three cases for $t_1^*(x_1)$.

<u>Case 1</u>:

$$t_1^*(x_1) = 0 \Rightarrow F(v_1) = 0 \ .$$

Then $C_1(x_1,v_1)$ may be written in equivalent form as

$$C_1(x_1,v_1) = kx_1^a e^{-rt_1^*(x_1)} + p \int_{F(v_1)}^{t_1^*(x_1)} (D(y) - v_1,0)e^{-ry}dy \; .$$

Case 2:

$$t_1^*(x_1) = F\left(v_1 + w^{1-a}x_1^a\right) > F(v_1) \; .$$

Now, for the integrand of the first integral:

$$0 \leq y \leq F(v_1) \, ,$$

$$D(0) \leq D(y) \leq v_1 \, ,$$

or

$$D(y) - v_1 \leq 0 \; .$$

Hence $\max\,(D(y) - v_1,0) = 0$.

And, for the second integrand

$$F(v_1) \leq y \leq F\left(v_1 + w^{1-a}x_1^a\right) \, ,$$

$$v_1 \leq D(y) \, ,$$

or

$$0 \leq D(y) - v_1 \, .$$

Hence $\max\,(D(y) - v_1,0) = D(y) - v_1$.

Again $C_1(x_1,v_1)$ may be written as above in Case 1.

Case 3:

$$t_1^*(x_1) = t_2^*(x_2) \; .$$

Again as before, for $y \in [0,F(v_1)]$, $\max (D(y) - v_1,0) = 0$, and for $y \in \left[F(v_1),t_2^*(x_2)\right]$, $\max (D(y) - v_1,0) = D(y) - v_1$.

Hence without loss of generality we can write

$$(14) \quad C_1(x_1,v_1) = kx_1^a e^{-rt_1^*(x_1)} + p \int_{F(v_1)}^{t_1^*(x_1)} (D(y) - v_1)e^{-ry}dy$$

where $F(v_1) = 0$ whenever $v_1 \leqq D(0)$.

Remark:

Case 3, $t_1^*(x_1) = t_2^*(x_2)$ is a degenerate case but presents no real difficulty. Whenever two-stages "collapse" into one stage, the cost function for a single stage is less than for two stages, for expansions at the same point in time. As an example, for $v_2 = v_1 + x_1$ and $t_1 = t_2$:

$$C(x_1,v_1) + C(x_2,v_1 + x_1) > C(x_1 + x_2,v_1)$$

because

$$\left(kx_1^a + kx_2^a\right) > k(x_1 + x_2)^a$$

for $0 < a < 1$.

Now, consider the second stage cost function, and, as before examine the last term. If $t_2^*(x_2) = t_1^*(x_1)$ the last term vanishes. If $t_2^*(x_2) = t_3^*(x_3)$ the case is degenerate and three stages have collapsed into two stages. However, if $t_2^*(x_2) = F\left(v_2 + w^{1-a}x_2^a\right)$ then the arguements similar to those leading to (14) can be made. Write the integral as follows:

$$p \int_{t_1^*(x_1)}^{t_2^*(x_2)} \max\,(D(y) - v_2, 0) e^{-ry} dy = p \int_{t_1^*(x_1)}^{F(v_2)} \max\,(D(y) - v_2, 0) e^{-ry} dy$$

$$+ p \int_{F(v_2)}^{t_2^*(x_2)} \max\,(D(y) - v_2, 0) e^{-ry} dy \ ,$$

where $v_2 = v_1 + x_1$ and $F(v_2) = F(v_1 + x_1)$. Then, the first integral always vanishes because

$$t_1^*(x_1) \leqq y \leqq F(v_1 + x_1)$$

and

$$D(y) \leqq v_1 + x_1 \ ,$$

or

$$D(y) - v_1 - x_1 \leqq 0 \, .$$

Hence

$$\max\,(D(y) - v_2, 0) = 0 \ .$$

Also, for the second integral,

$$F(v_2) = F(v_1 + x_1) \leqq y \leqq t_2^*(x_2) \ ,$$

or

$$v_1 + x_1 \leqq D(y) \, .$$

Hence:

$$\max\ (D(y) - v_2, 0) = D(y) - v_2 \ .$$

Therefore, the cost function for the second stage can be written

$$C_2(x_2, v_2) = kx_2^a e^{-rt_2^*(x_2)} + p \int_{F(v_2)}^{t_2^*(x_2)} (D(y) - v_2)e^{-ry}dy \ ,$$

where $t_2^*(x_2)$ is defined as before. In general, then, we can write for the jth stage,

$$(15) \qquad C_j(x_j, v_j) = kx_j^a e^{-rt_j^*(x_j)} + p \int_{F(v_j)}^{t_j^*(x_j)} (D(y) - v_j)e^{-ry}dy \ ,$$

where: $v_j = v_1 + x_1 + \ldots + x_{j-1}$, $\quad t_o^*(x_o) \equiv 0$, and

$$(16) \qquad t_j^*(x_j) = \left\{ \begin{array}{ll} t_{j+1}^*(x_{j+1}) & \text{if } x_j \in [0,w] \\ t_{j-1}^*(x_{j-1}) & \text{if } x_j \in (w,z) \text{ and } w + v_j < D(0) \\ F\left(v_j + w^{1-a}x_j^a\right) & \text{otherwise} \end{array} \right\} .$$

<u>Computational Procedure, Type I Problems</u>

1. Divide the capacity and demand interval $[0,D(I)]$ into M subintervals. For illustrative purposes only, assume these subintervals are of uniform size, say $\Delta > 0$. Consider v_1 , the initial capacity to be a parameter instead of a fixed value, with $v_1 \in [v_o, D(I)]$ where v_o and $D(I)$ are fixed values. Restrict positive values of x_j , $D(\cdot)$, v_j to multiples of Δ . Hence , for a j-stage problem,

$$v_j + x_j = D(i) \; \varepsilon \; \{0,\Delta,2\Delta, \ldots, M\Delta\}$$

and

$$v_j \; \varepsilon \; \{0,\Delta,2\Delta, \ldots, M\Delta\}$$

(Note: except for linear demand, F($\cdot$) is not restricted to multiples of some value.)

2. Choose a value for D(i) and a value for v_1.
3. Compute $t_1^*(x_1)$ from (13).
4. Compute $C_1(x_1,v_1)$ from (14).
5. Repeat Steps 2,3,and 4 for all $D(i) \; \varepsilon \; [v_o,D(I)]$ and all $v_1 \varepsilon [v_o,D(I)]$.
6. Store the results in Table I. The subscript j is used because Table I is used for any succeeding single stage cost function.

$C_j(\,D(i),v_j)$ $\quad j=1,2,\ldots,n$							
D(i) / v_j	v_o	$v_o + \Delta$	$v_o+2\Delta$	....	D(I)-2Δ	D(I)-Δ	D(I)
v_o			*				
$v_o+\Delta$							
$v_o+2\Delta$							**
⋮							
D(I) - 2Δ							
D(I) - Δ							
D(I)							

TABLE I

(Note: diagonal entries are zero and all entries below the diagonal are also zero).

7. Now, consider a two stage problem. The first stage *end* conditions are the second stage *initial* conditions. Therefore, demand equals capacity at the beginning of the second stage, except when $x_1 = 0$ or $x_1 = D(I)-v_o$, in which case there is no second stage. Hence both first and second stage cost functions are found in Table I. For example, suppose $x_1 = 2\Delta > w$ and $v_1 = v_o$ for the first stage, and, $x_2 = D(I) - 2\Delta - v_o$, $v_2 = v_o + 2\Delta$ for the second stage. Then, the entry in Table I for the second stage cost function, corresponds to column D(I) and row $(v_o + 2\Delta)$. For this example, the asterisk in Table I is for the first stage and the double asterisk for the second stage.
8. Restrict the sum of x_1 and x_2 such that

$$x_1 + x_2 = [\ D(I) - v_o\ ].$$

Let $v_1 = v_o$. Then, since

$$v_2 = v_o + x_1,$$

Table I can be used to compute by enumeration:

$$C_2^*(D(I)) = \underset{v_2 \varepsilon [v_o, D(I)]}{\text{minimum}} \{\ C_2(D(I), v_2) + C_1^*(D(I) - v_2)\ \},$$

where $C_1^*(D(I))$ is the optimal solution for a one stage problem. This yields the optimal policy for a capacity expansion of size $D(I)-v_o$ done in n=2 stages. But if expansion in two stages is not as good as expansion in one stage, $x_2^* = 0$. Hence, this also yields the best value for n^*, the optimal number of stages.
9. Now, consider a three stage problem, with a third stage initial capacity of v_3, where:

$$v_3 = v_o + x_1 + x_2\ .$$

For a given value of v_3 , the optimal total cost function is given by the sum of the third stage cost function plus an optimal two stage cost function, i.e.,

$$C_3(D(I),v_3) + C_2^*(D(I) -v_3).$$

We wish to minimize this sum by choosing the best value for v_3. Hence, compute by enumeration, $C_2^*(D(I) - v_3)$ for all $v_3 \varepsilon [v_o, D(I)]$ and store in Table II.

v_3	$C_2^*(D(I)-v_3)$
0	
Δ	
⋮	
D(I)	

TABLE II

10. Now, use Table I and II to compute by enumeration:

$$C_3^*(D(I)) = \underset{v_3 \ \varepsilon \ [v_o,D(I)]}{\text{minimum}} \{ C_3(D(I),v_3) + C_2^*(D(I) -v_3)\} .$$

This yields the optimal policy, including n^*, for an n=3 stage problem and if expansion in three stages is not as good as two stages, $x_3^* = 0$. (If expansion in three stages is not as good as one stage, $x_3^* = x_2^* = 0$.)

11. Repeat step 9 and 10, adding a fourth stage and construct a Table III of values for $C_4^*(D(I)-v_4)$, using Tables I and II. Discard Table II if desired.
12. Terminate the process when the previous table is repeated. If this happens for n=k+1, then $n^* = k$.

Remark:

We can now write the general recursion relation:

$$C_n^*(D(I)) = \underset{v_n \varepsilon [v_o, D(I)]}{\text{minimum}} \left\{ C_{n-1}^*(D(I) - v_n) + C_n(D(I), v_n) \right\} . \tag{20}$$

Type II Problems

Consider now a Type II problem in which the shortage after the last expansion is permitted. Denote the shortage after the nth capacity expansion as:

$$S(v_n) = p \int_{F(v_n)}^{I} (D(y) - v_n) e^{-ry} dy.$$

This function is the no expansion case whenever $v_n = v_o$.

Denote $T_2(\cdot)$ as a two stage Type II cost function. For a given value of v_2, the two stage cost function may be written as the sum of the shortage cost plus an optimal n =2 stage Type I cost function:

$$T_2(v_2) = C_2^*(v_2) + S(v_2) \quad .$$

We wish to minimize this by choosing the best value for v_2. Hence, we define

$$T_2^*(D(I)) = \underset{v_2 \,\varepsilon\, [v_o, D(I)]}{\text{minimum}} \left\{ C_2^*(v_2) + S(v_2) \right\}$$

as the optimal cost function for a two stage, Type II capacity expansion problem. Similarly, for an n-stage problem we have:

$$T_n^*(D(I)) = \underset{v_n \,\varepsilon\, [v_o, D(I)]}{\text{minimum}} \left\{ C^*_n (v_n) + S(v_n) \right\}.$$

The Type II computational procedure parallels the Type I procedure after a table of values for $S(v_1)$ has been computed for all $v_1 \,\varepsilon\, [v_o, D(I)]$.

(Note that the no expansion policy is not found by Type I computations but it is found by Type II computations.)

CHAPTER V

A STATIONARY POLICY FOR A LINEAR DEMAND FUNCTION

Introduction

In this chapter we use our previous results for a finite sequence of capacity expansions and argue an asymptotic stationary model as a limiting case for a linear demand function, with a planning interval of infinite length and an infinite number of stages. Manne [9] who first investigated this problem uses the notion of "regeneration points" to derive a cost function.

Consider a sequential capacity expansion problem in which the demand function is linear and the initial demand as well as initial capacity are zero. Let the rate of growth of demand be "b" . Then

$$D(y) = by \quad \text{and} \quad D(0) = v = 0$$

$$F(x) = \frac{1}{b} x \ .$$

From Equation (16), Chapter IV we know that for $v_o = D(0) = 0$, the optimal time of expansion for the nth stage, $n = 1,2, \ldots, N$, reduces to two cases:

$$t_n^*(x_n) = \begin{Bmatrix} t_{n+1}^*(x_{n+1}) & \text{if} \quad 0 \leqq x_n \leqq w \\ F\left(v_n + w^{1-a} x_n^a\right) & \text{otherwise} \end{Bmatrix}$$

where

$$v_n = v_1 + \sum_{i=0}^{n-1} x_i \ .$$

Also from Chapter IV we know the configuration of the optimal expansion policy is $F(x_1 + x_2 + \ldots + x_{n-1}) \leq t_n^*(x_n) \leq F(x_1 + \ldots + x_n)$ because $v_1 = 0$. Hence we may write the cost function for an n-stage Type I problem as follows. Let: $x_o = t_o = 0$ and $t_{n+1} = I$. For brevity denote:

$$C_n = C_1(x_1,v_1) + C_2(x_2,v_2) + \ldots + C_n(x_n,v_n) .$$

Then:

$$C_n = K \sum_{i=1}^{n} x_i^a e^{-rt_i} + p \sum_{i=1}^{n} \int_{F(v_i)}^{t_i} (by - v_i)e^{-ry}dy .$$

Define a stationary policy as one in which all capacity expansions are of equal magnitude. This is equivalent to specifying that the time intervals between expansions are equal for an optimal policy. To see this consider that the time interval between the nth and n + 1th expansions, is:

$$t_{n+1}^* - t_n^* = F\left(\sum_{i=0}^{n} x_i + w^{1-a}x_{n+1}^a\right) - F\left(\sum_{i=0}^{n-1} x_i + w^{1-a}x_n^a\right) .$$

And:

$$x_1 = x_2 = \ldots = x_n = x_{n+1} = x \Rightarrow F\left(\sum_{i=1}^{n} x_i\right) = nF(x) .$$

Hence

$$t^*_{n+1} - t^*_n = F(nx + w^{1-a}x^a) - F((n-1)x + w^{1-a}x^a)$$

$$= \frac{1}{b}(nx + w^{1-a}x^a) - \frac{1}{b}((n-1)x + w^{1-a}x^a)$$

$$= \frac{1}{b}(x + w^{1-a}x^a)$$

and the time intervals between successive expansions is constant. In particular:

$$t^*_1(x) = \frac{1}{b} w^{1-a}x^a$$

$$t^*_2(x) = F(x) + t^*_1(x)$$

and

$$t^*_i(x) = (i-1)F(x) + t^*_1(x) \; ; \qquad i = 1,2, \ldots, n \; .$$

The asymptotic cost function as $n \to \infty$ and $I \to \infty$ is denoted C_∞ and using the preceding we may write it as follows.

$$C_\infty = \sum_{i=1}^{\infty} kx^a e^{-rt_i} + \sum_{i=0}^{\infty} \left\{ p \int_{iF(x)}^{iF(x)+t_1} (by - ix)e^{-ry} dy \right\} .$$

Note:

$$e^{-rt_{i+1}} = \left[e^{-r(iF(x)+t_1)}\right] = \left[e^{-rF(x)}\right]^i e^{-rt_1}$$

hence

$$\sum_{i=1}^{\infty} kx^a e^{-rt_i} = \left(kx^a e^{-rt_1}\right)\left\{1 + e^{-rF(x)} + \left[e^{-rF(x)}\right]^2 + \ldots \right\}$$

$$= \left(kx^a e^{-rt_1}\right) \sum_{i=0}^{\infty} \left[e^{-rF(x)}\right]^i$$

$$= \frac{kx^a e^{-rt_1}}{1 - e^{-rF(x)}}$$

because $\left|e^{-rF(x)}\right| < 1$ for $x > 0$. Next examine the second term, the integral, in C_∞.

$$p\int_{iF(x)}^{iF(x)+t_1} bye^{-ry}dy - pix\int_{iF(x)}^{iF(x)+t_1} e^{-ry}dy$$

$$= \frac{pb}{r^2}\, e^{-riF(x)}\left\{\left[b + briF(x) - rix\right] + e^{-rt_1}\left(rix - b - briF(x) - brt_1\right)\right\}.$$

To compare this result directly with the asymptotic stationary model of Manne [9], we normalize the time scale, letting $b = 1$ and, therefore, $F(x) = x$. Then the above expression becomes

$$= \frac{p}{r^2}\, e^{-rix}\left[1-e^{-rt_1}\left(1 + rt_1\right)\right].$$

Then

$$C_\infty = \frac{kx^a\, e^{-rt_1}}{1 - e^{-rx}} + \frac{p}{r^2}\left(1-e^{-rt_1}\left(1 + rt_1\right)\right)\sum_{i=0}^{\infty}\left[e^{-rx}\right]^i$$

Again, noting that $\left|e^{-rx}\right| < 1$ for $x > 0$, we can write

$$\sum_{i=0}^{\infty}\left[e^{-rx}\right]^i = \frac{1}{1 - e^{-rx}}$$

Hence

$$C_\infty = \left[\frac{1}{1 - e^{-rx}}\right]\left[kx^a e^{-rt_1} + \frac{p}{r^2}\left(\left(1 - e^{-rt_1}\left(1 + rt_1\right)\right)\right)\right].$$

This result agrees with Manne [9]. It can be simplified by noting that

$$t_1^*(x) = \frac{rk}{p} x^a$$

or

$$\frac{p}{r} t_1^*(x) = kx^a \ .$$

Substituting this into the above yields a cost function in one variable, x :

$$C_\infty(x) = \frac{p}{r^2} \left(\frac{1 - e^{-rt_1^*(x)}}{1 - e^{-rx}} \right) .$$

This is the same result as that found by Erlenkotter in Chapter 11 of Manne [10]. There the derivation was based on the notion of "regeneration points" rather than directly arguing the case as an asymptotic result of a sequential capacity expansion model.

In the chapter which follows, we will use the algorithm of Chapter IV to compare a finite planning interval policy with a stationary policy.

Remarks:

There are distinations between assuming an infinite planning horizon and a finite planning horizon. One reason Manne suggests an infinite horizon is that he doubts that geometric growth rates which characterize some industries today "can be extrapolated to the indefinite future." Another reason he gives is that his cost function is mathematically tractable. Finally, his model sheds light on the question of optimal policies for *standard sized plants*. However, the chances of correctly forecasting a demand function *far* into the future are very small indeed, hence a policy based on such deterministic forecasts is not truly optimal. On the other hand, a policy based solely on short term forecasts may not be truly optimal either.

Although it may be possible to assign some probabilities to one's demand forecasts and attempt to minimize an expected present value function,

it is likely to prove very cumbersome to use. In practice, planners are more likely to use a deterministic model and treat demand forecasts in a parametric fashion.

There usually is enough time between expansions to permit forecasts to be *up-dated* and new determinations made of optimal policies regardless of which model is used. Whatever the length at the planning horizon, if demand is not a linear function, a stationary policy may be a poor approximation to reality. For this last reason, if for no other, we feel our approach yields results which may be of practical importance.

CHAPTER VI

SOME COMPUTATIONAL RESULTS

Introduction

The discrete dynamic programming algorithm for Type I problems was coded in Fortran IV for the Univac 1100 computer. Computations were made for seven demand functions.

The first set of demand functions, shown in Figure 8 , were chosen to facilitate comparing a stationary policy with a multi-stage finite planning interval policy.

The second set of demand functions, shown in Figure 16, were chosen to test the sensitivity of x_1^* , the optimal value for the first stage capacity expansion, to changes in demand forecast, using a family of demand functions (all of which coincide during the first third of the planning interval).

The following were chosen so as to simplify comparisons with the asymptotic case:

$$a = .5 \ , \ k = 8 \ , \ p = 1 \ , \ r = 0.1 \ , \ I = 60 \ .$$

All computations employed the same length planning interval, namely 60 units. The units of measure for the planning interval are arbitrary. The demand functions for Cases A, B, C were: linear, strictly concave and strictly convex, with $D(0) = 0$ and $D(I) = 60$ in all three cases see Figure 8.

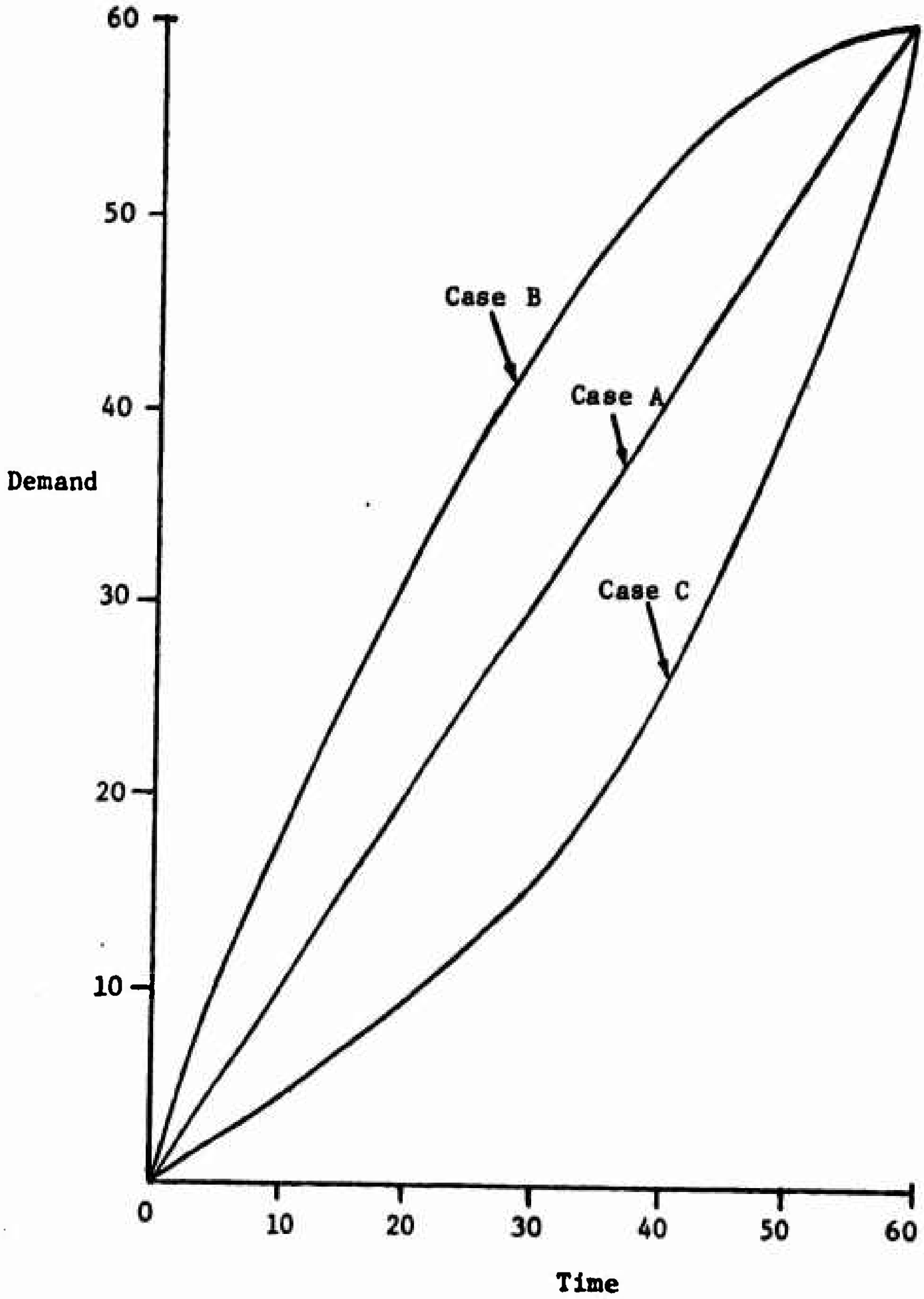
60
50
40
30
20
10
Demand
Case B
Case A
Case C
0
10
20
30
40
50
60
Time

FIGURE 8

Figure 9 shows the present value of total costs as a function of the planning interval. The value of n^* changes at 20, 36, and 51 periods.

<u>Run A: Linear Demand Function</u>

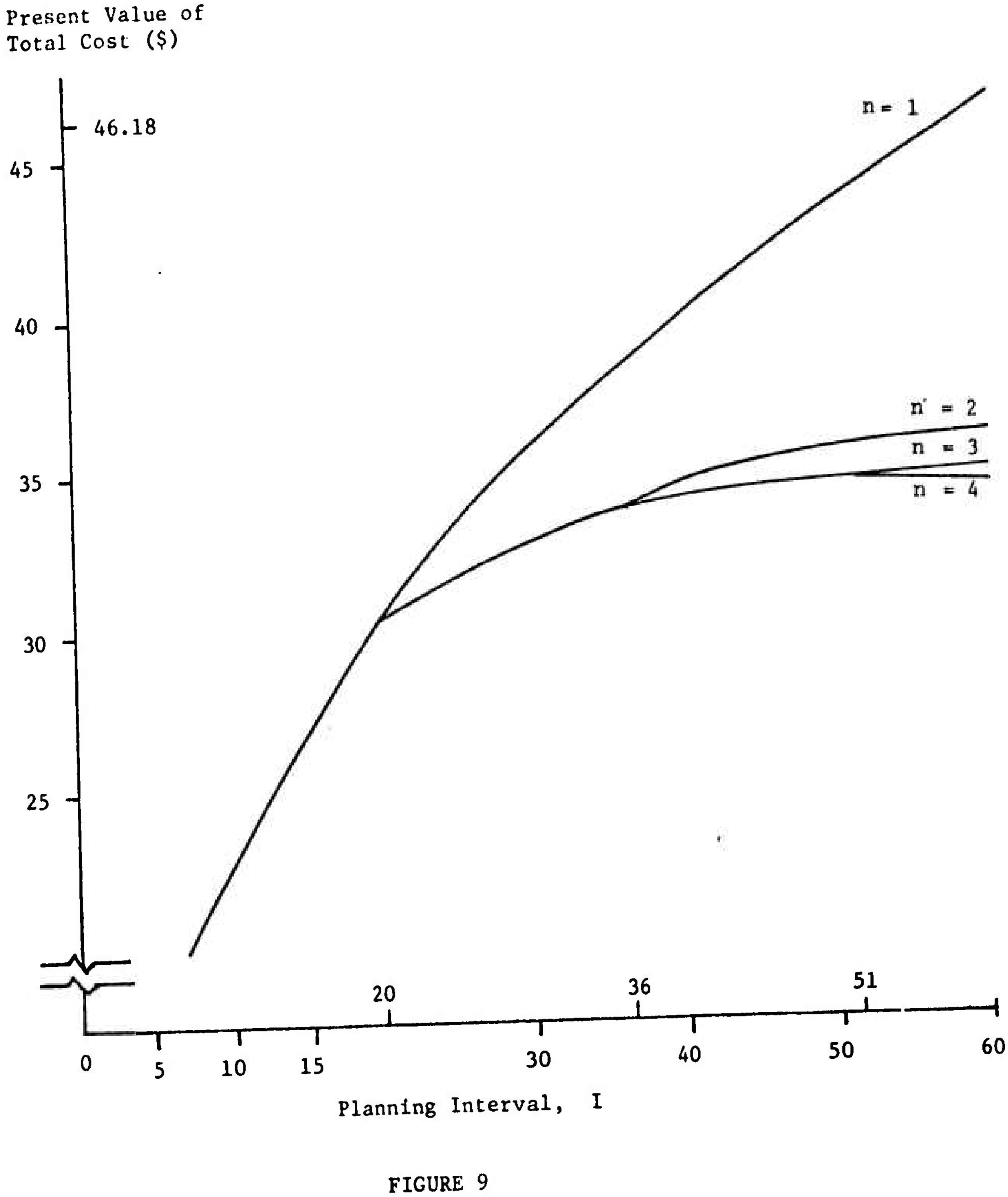

FIGURE 9

Figure 10 is the lower portion of Figure 9 and is drawn to the same scale. It shows a comparison of a no-expansion policy with the policy for $n = 1$ and $n = 2$ expansions. For a planning interval less than 8.7 periods (approximately), the no-expansion policy is optimal.

Case A: Linear Demand Function

Comparison of No Expansion with an Optimal n=1,2 Expansion Policy

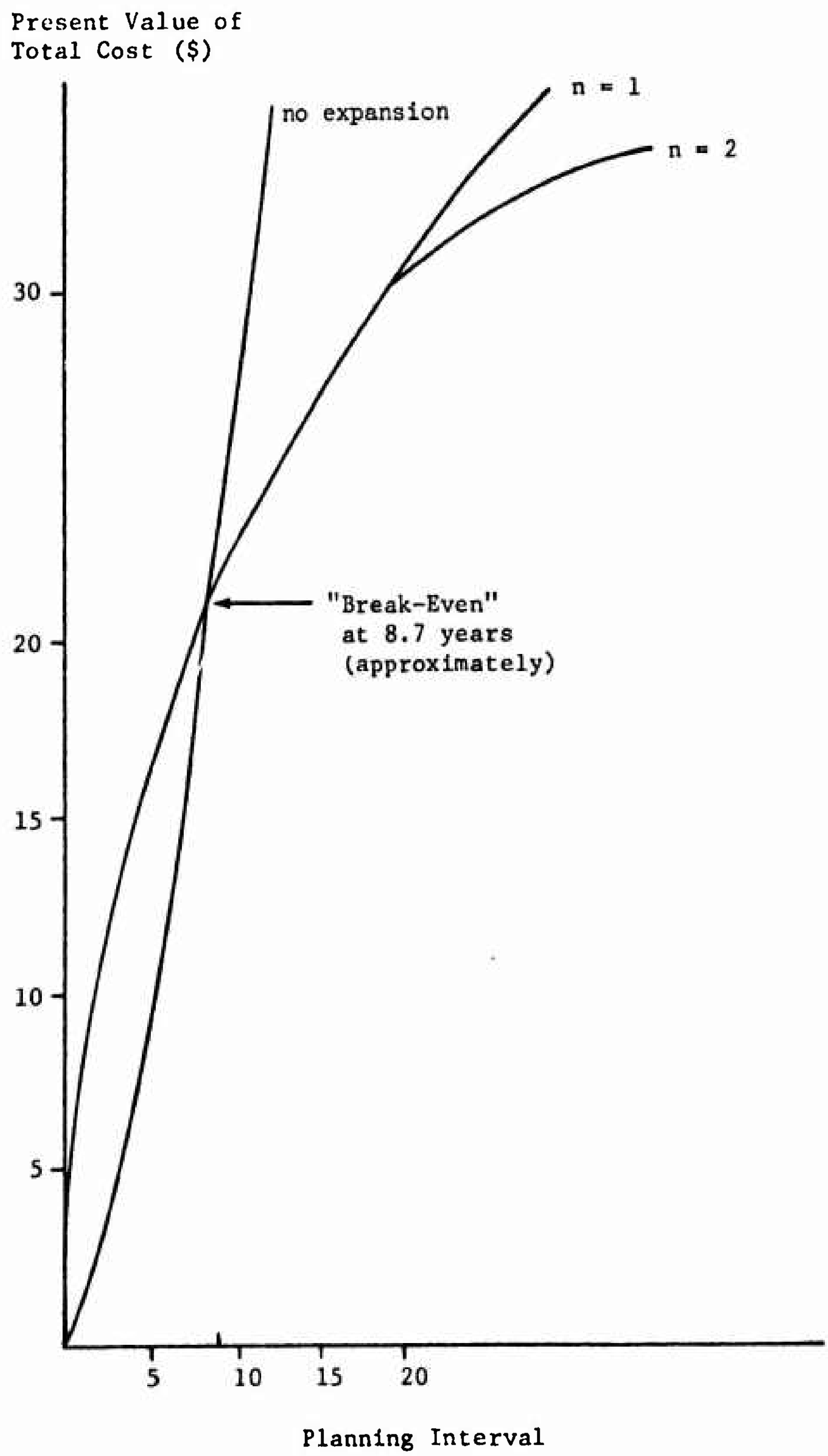

FIGURE 10

Case B: Concave Demand Function

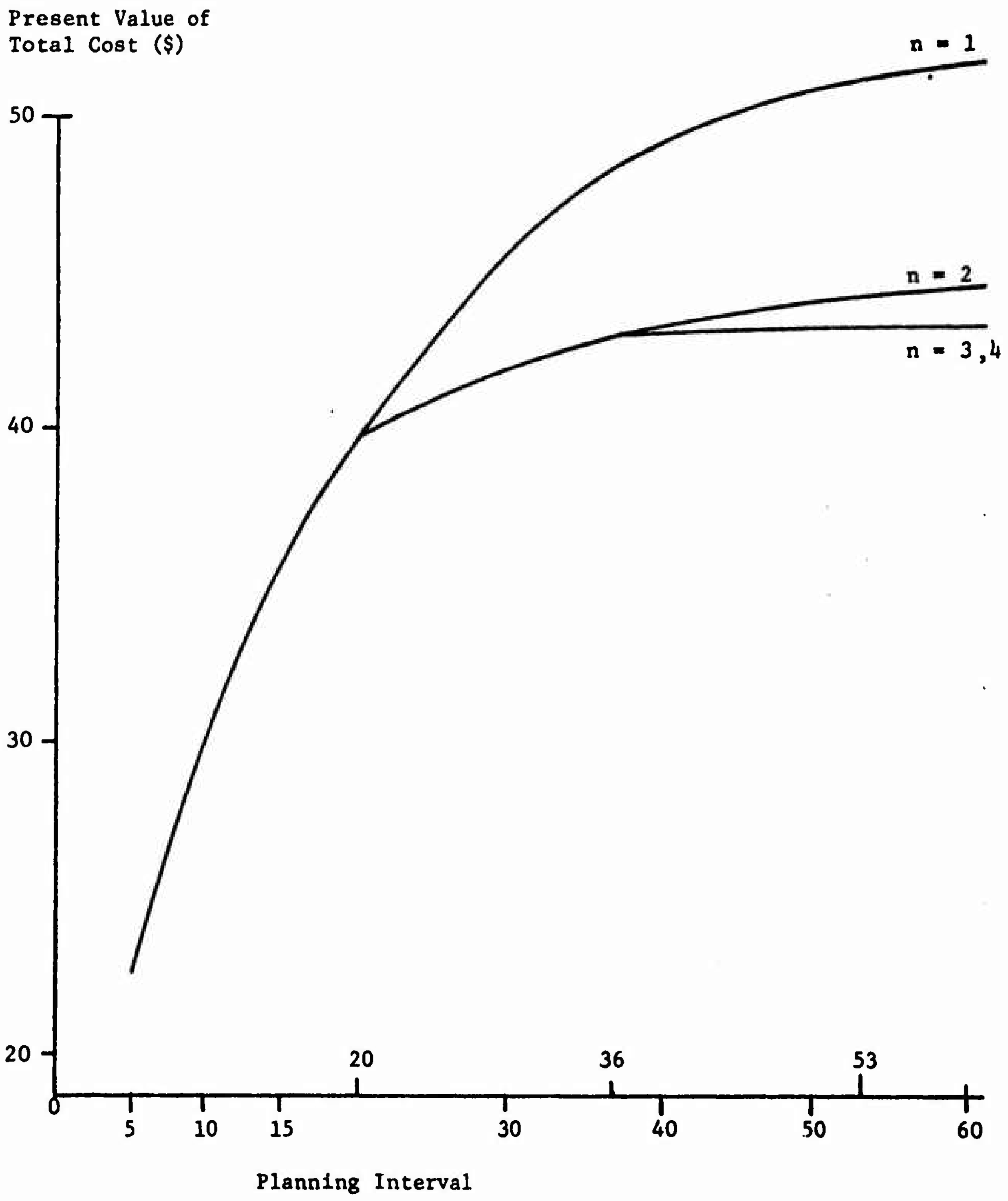

FIGURE 11

Case C: Convex Demand Function

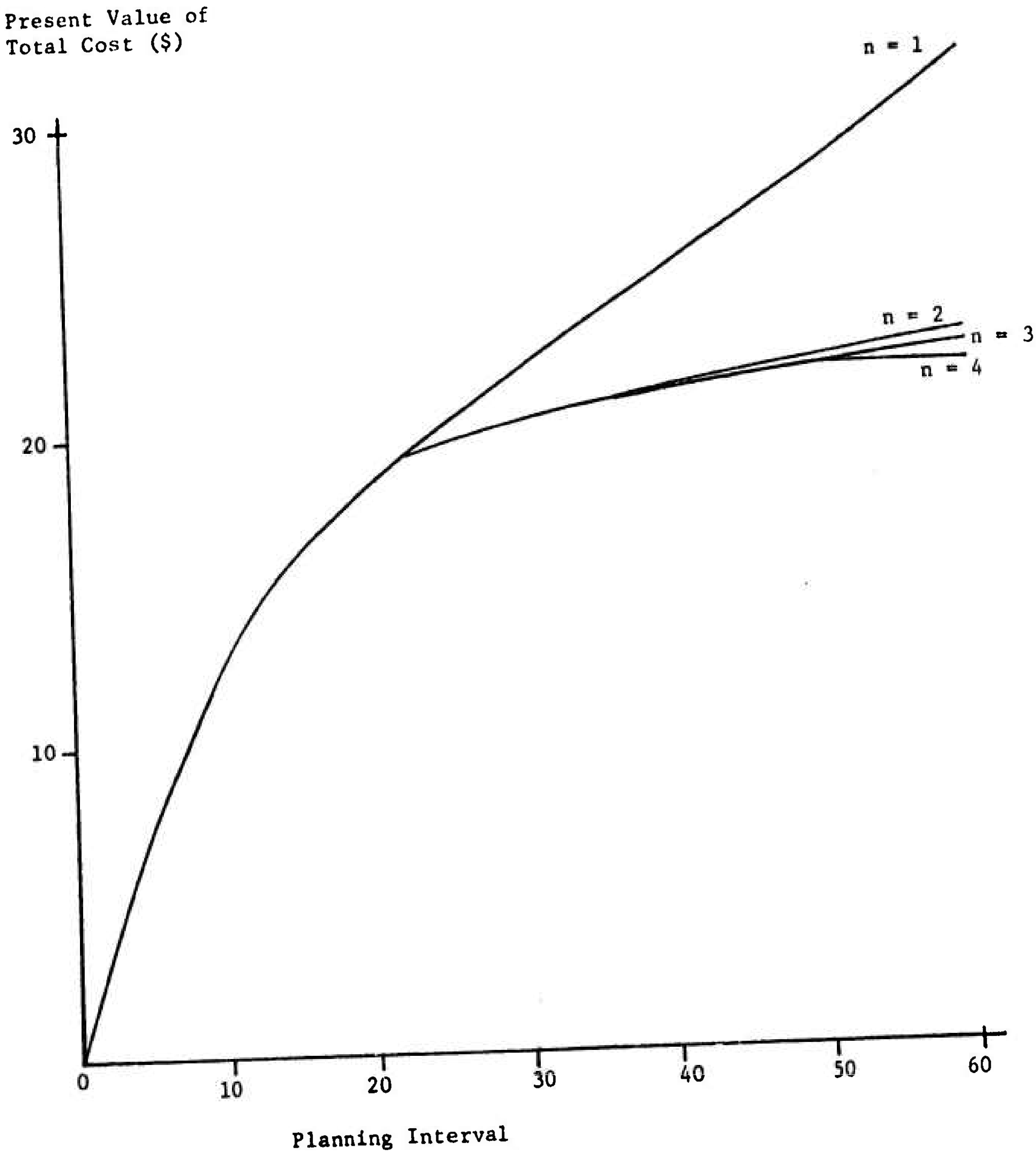

FIGURE 12

Discussion of Computational Results for Cases A, B and C

The computer code permits a maximum of seven expansions. In Cases A, B and C, the optimal number of expansions, n^*, was four. We note that the optimal value of the cost function decreases with the number of expansions up to n^* and is constant for all $n \geq n^*$.

The point in time at which a two stage policy first becomes optimal is almost the same for all three trials: 20 periods for concave and linear demand functions, 22 periods for the convex demand function. Similarly, after 36 periods a three stage policy is optimal for Case A, B and C.

Case A: Linear Demand Function

In Manne [9], Chapter 10, Appendix B, the following results are given for a linear demand function $D(y) = y$ using the asymptotic stationary policy cost function:

$$x^* = 15.17$$

$$t_1^* = 3.12$$

$$\frac{C_\infty}{K} = 4.287 .$$

The graphs of Figure 13, showing the optimal size of expansions as a function of the length of the planning interval illustrates three points in time for which the optimal policy is one of equal expansions -- 30.34, 45.51 and 60.68 periods. These points coincide with the "regeneration" points of the stationary model.

Also note that a rather long planning interval must be used before a two-expansion policy is optimal (21 units) and that smaller planning intervals may imply nonequally-sized expansions.

For planning intervals greater than 21 periods (when the two-stage policy becomes optimal), the optimal size of the first expansion varies only slightly from the stationary value of 15.17. Similarly, after 38 periods the second expansion varies only slightly from its stationary value of 15.17. Presumeably similar remarks would apply for the third expansion if the planning interval were extended beyond 60.68 periods. Moreover, the optimal number of stages for 60 periods is four, the same as for the stationary model. These findings tend to confirm a principal hypothesis of a stationary model, namely, that the optimal policy is one in which the expansions are of equal size.

Optimal Size of Expansion

In this section we consider the optimal size of each expansion as a function of the planning interval. The computational results are in graphical form. (See Figures 13, 14 and 15).

Case A: Linear Demand

Optimal Size of Expansion as a Function of the Planning Interval Length

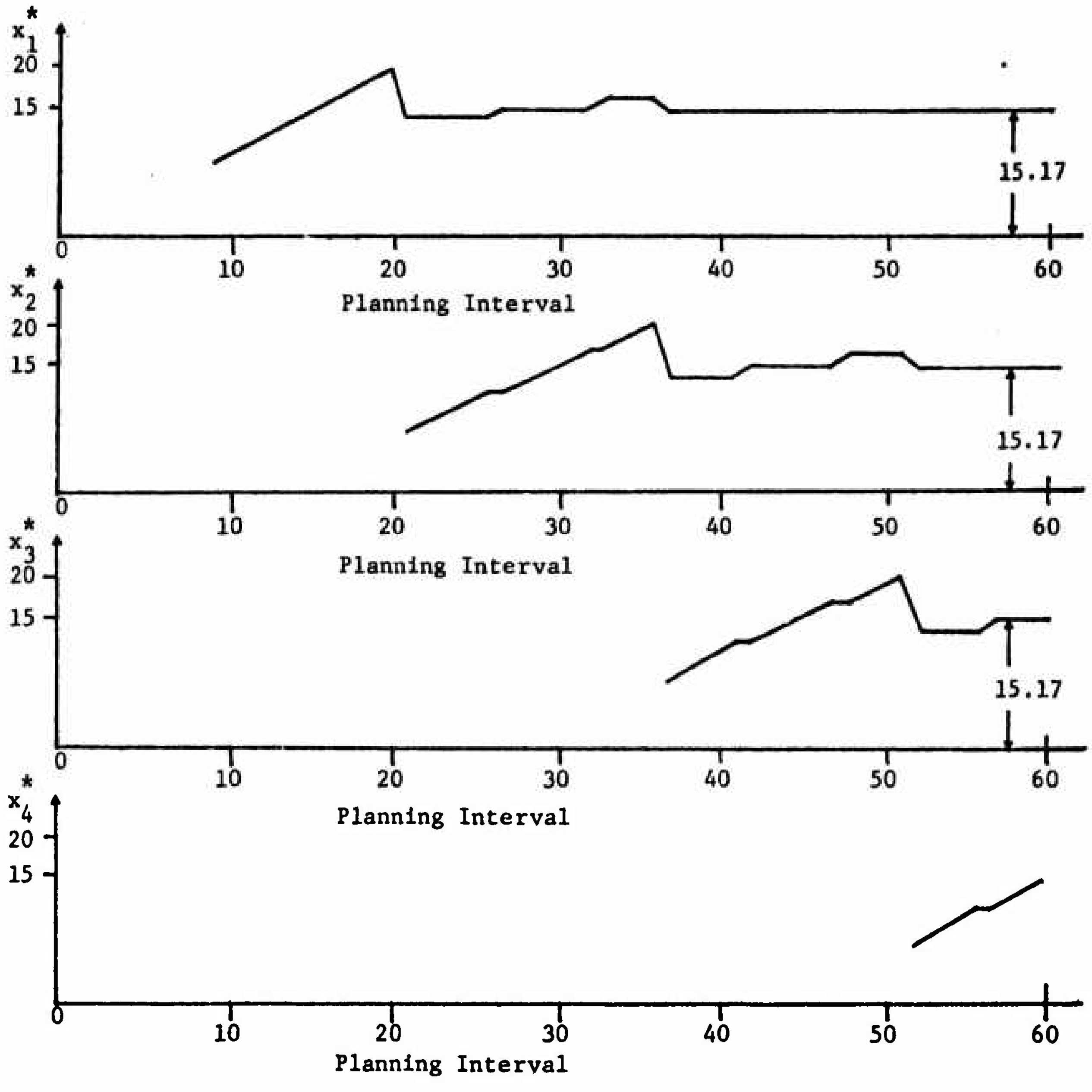

FIGURE 13

Since only discrete values were computed, the continuous curves were drawn for ease of reading. The maximum value of x_1^*, x_2^*, x_3^* is 20. The stationary value of x^*, using the stationary model, is 15.17.

In the first graph above, the optimal policy is $x_1^* = 0$ for a planning interval less than approximately 8.7 units of time.

Case B: Concave Demand

Optimal Size of Expansion as a Function of the Planning Interval Length

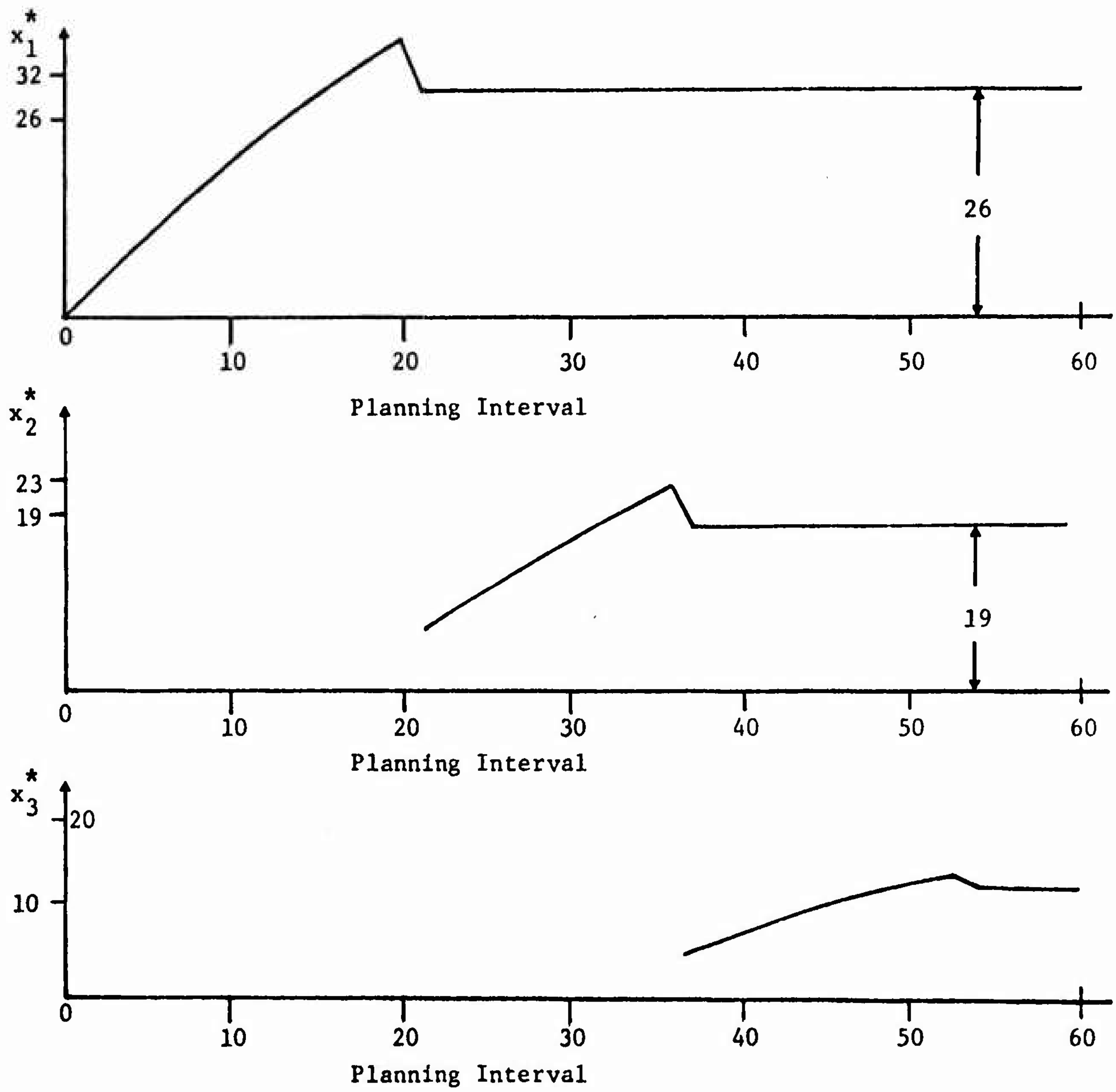

FIGURE 14

Notes:

No graph is shown for x_4^* because it consists of only one point at $I = 60$.

The no-expansion policy was *not* computed; hence, the graphs are drawn as if $x_1^* > 0$ even for small planning intervals.

Case C: Convex Demand

Optimal Size of Expansion as a Function of the Planning Interval Length

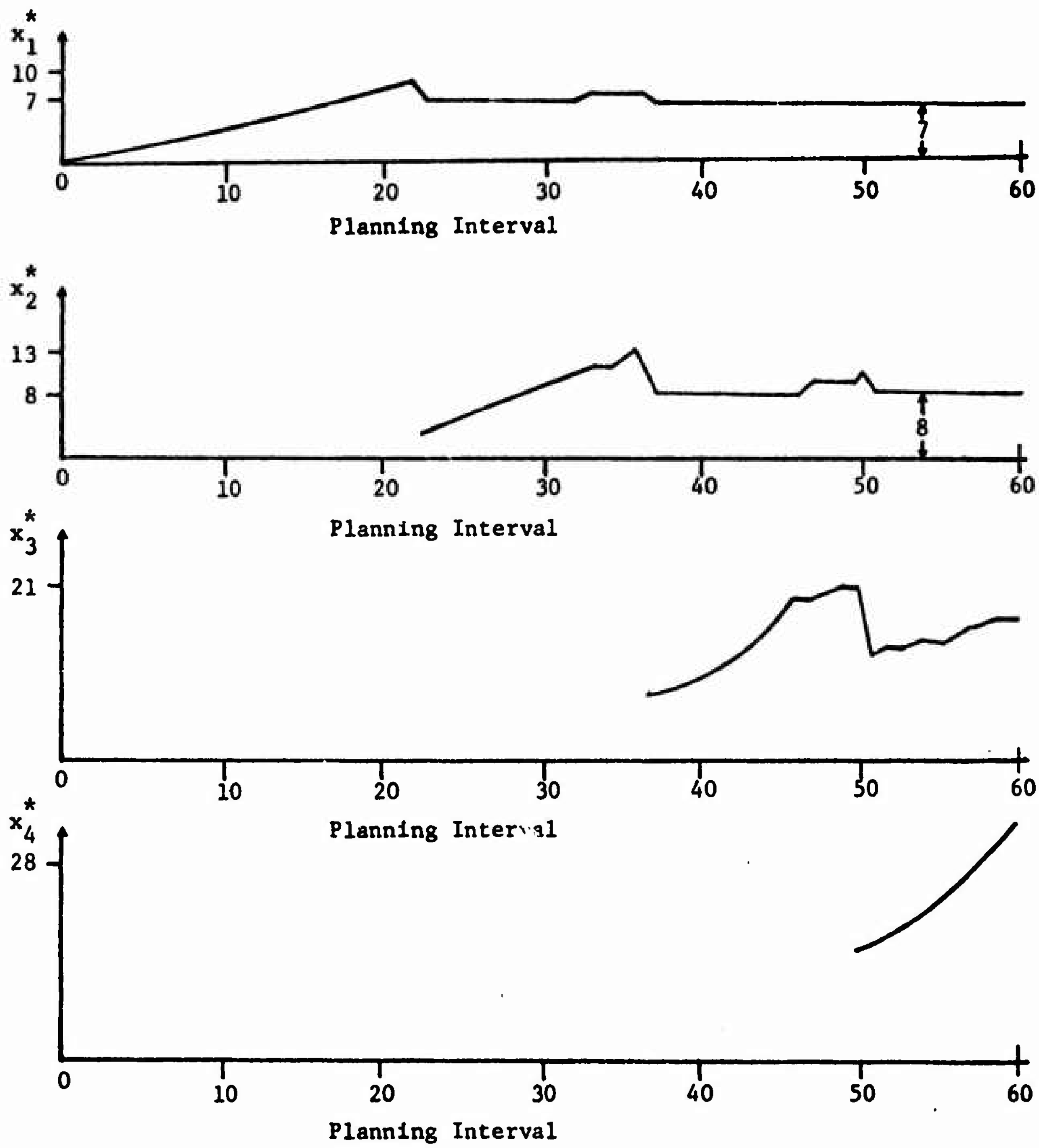

FIGURE 15

Note:

The no-expansion policy was *not* computed; hence, graphs are drawn as if the optimal policy is $x_1^* > 0$ even for small planning intervals.

One result is common for all three demand functions, namely that x_1^*, x_2^* the sizes of the first and second expansions tend to fluctuate around some constant value. And, this value is reached at a point in time just after the optimal number of stages increases from one to two for x_1^* and from two to three for x_2^*. This suggests that, for example, the value of x_1^* may be invariant with respect to a family of demand functions which are identical near the origin. This portion is usually the most reliable part of a demand forecast. Accordingly, we tested this idea using Case C to define three additional demand functions denoted Cases: C1, C2 and C3. These four demand functions are identical for $I \epsilon [0,20]$ only. (See Figure 16.) These functions were deliberately chosen to provide distinctively different patterns both as to rate of growth and final values for demand at the end of the planning interval.

A Family of Demand Functions

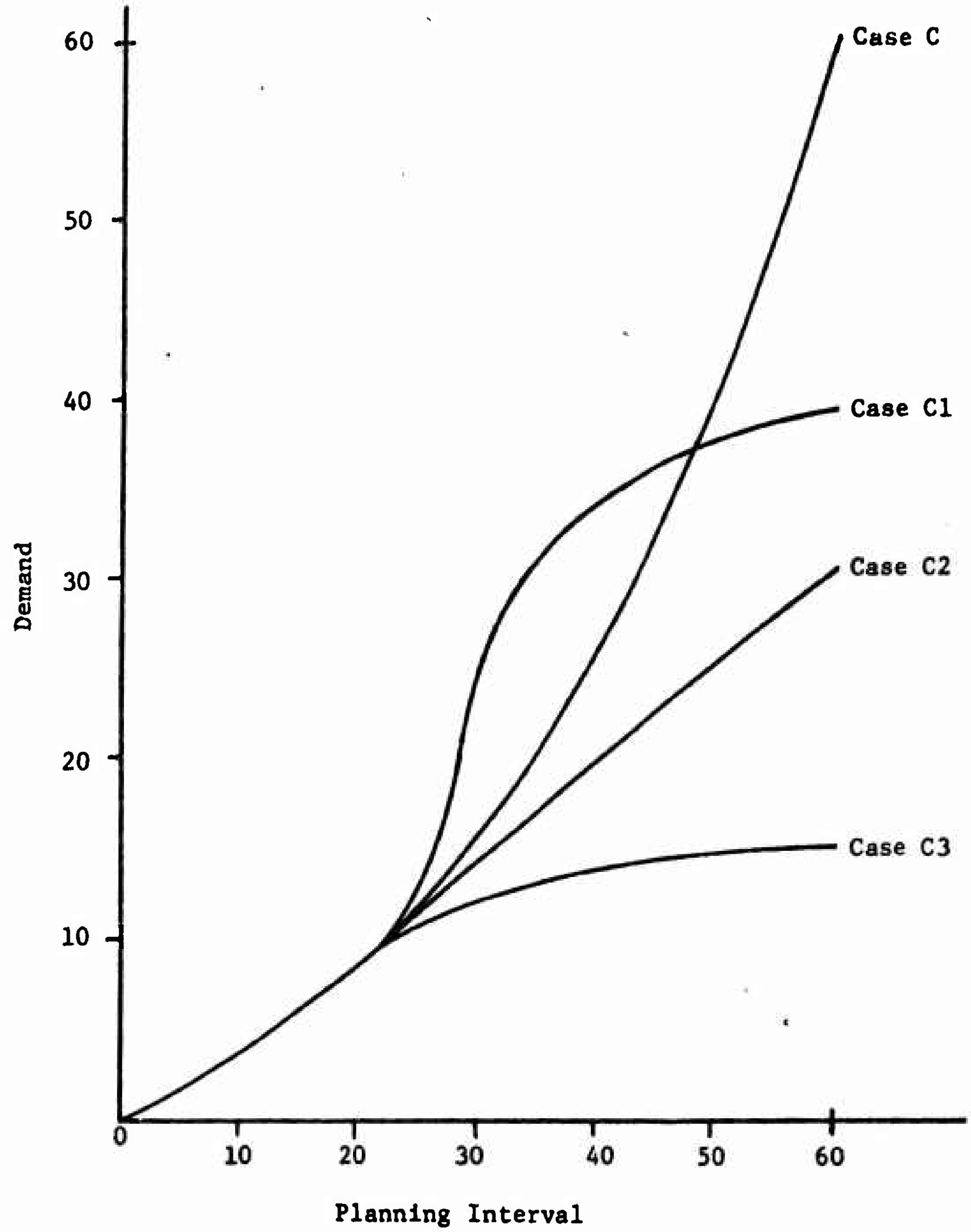

FIGURE 16

STABILITY OF x_1^*

Cases: C, C1, C2 and C3

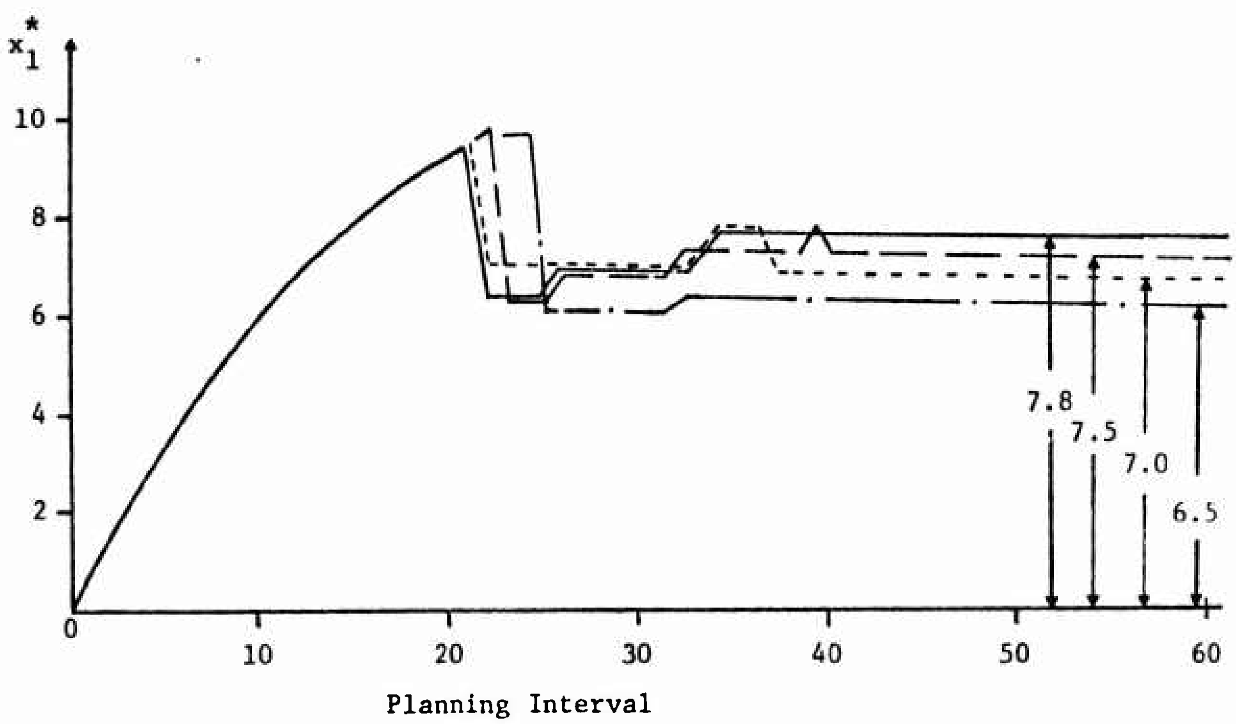

FIGURE 17

LEGEND

- - - - - Case C
——— Case C1
— — — Case C2
— · — Case C3

The results, in graphical form, are shown in Figure 17. Only x_1^* , the optimal value for the first expansion, is shown. (For our purposes we have assumed that the optimal policy is not a no-expansion policy.) As before, a two stage policy is optimal at about the same time, between 22 and 25 years. (This coincides with the first decrease in x_1^* .) In each case, x_1^* tends to a constant value and further, these constant values are close to one another.

The importance of this result is as follows. In practice one is most interested in ascertaining the optimal size of the first expansion because it is the first commitment. Subsequently, new demand forecasts and a new policy can be determined. It is generally true that the reliability of a demand forecast decreases with the length of the planning interval. This suggests treating demand forecasts parametrically. Our results lead us to conclude that if short term forecasts are reliable, good estimates can be made for the optimal size of the first expansion.

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